Extention of Topological Inner Invariant Means

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Abstract

For a locally compact group $G$, we prove that a topological inner invariant mean on $LUC(G)$ has an extension to a topological inner invariant mean on $L^\infty(G)$.

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1 Introduction

Let $G$ be a locally compact group with identity $e$ and left Haar measure $dx$. Let $L^\infty(G)$ and $L^1(G)$ denote the spaces of essentially bounded functions and integrable functions respectively.

For a function $f : G \to \mathbb{C}$ we put $x f(y) = f(xy)$ and $f_x(y) = f(yx)$ for all $x,y \in G$. Recall that a mean $m$ on a linear subspace $X$ of $L^\infty(G)$ containing the constants is a linear functional such that $||m|| = m(1) = 1$. If $x f, f_x \in X$ for all $f \in X$ and $x \in G$, we say that $m$ is inner invariant if $m(x f) = m(f x)$ for all $x \in G$ and $f \in X$.

We denote by $P(G)$ the set of all $\varphi \in L^1(G)$ with $\varphi \geq 0$ and $||\varphi||_1 = 1$. If $X$ is a linear subspace of $L^\infty(G)$ containing the constants, $X$ is said to be topological left (resp. right) invariant if $P(G) * X \subset X$ (resp. $X * P(G) \subset X$). Here $\tilde{\varphi}$ is the function defined by $\tilde{\varphi}(x) = \varphi(x^{-1})$ for all $x \in G$ and $*$ denotes the convolution product of functions on $G$. Let $X$ be topological invariant (i.e. topological left and right invariant). A mean defined on $X$ is said to be topological inner invariant if

$$m(\frac{1}{\Delta} \tilde{\varphi} * f) = m(f * \tilde{\varphi}) \text{ for all } f \in X \text{ and } \varphi \in P(G)$$
where $\Delta$ is the modular function of $G$. The concept of topological inner invariant means (TIIM) was introduced and studied by Nasr-Isfahani [8] for a large class of Banach algebras containing $L^1(G)$ known as Lau algebras.

In [7] authors study locally compact groups for which $L^\infty(G)$ has a TIIM whose restriction to $C_b(G)$ is not $\delta_e$. In this paper we show that a TIIM on $LUC(G)$ (space of left uniformly continuous functions on $G$) has a topological inner invariant extension to $L^\infty(G)$. Also based on the technic used in [7, Th.3.2] we give a simpler proof of the well known fact that amenability implies topological amenability (see [9]).

2 Topological inner invariant means

It is well known [7] that any extension $E$ of $\delta_e$ from $C_b(G)$ to a mean on $L^\infty(G)$ is a mixed identity of $L^\infty(G)^*$; that is

$$E(\frac{1}{\Delta} \tilde{\varphi} \ast f) = E(f \ast \tilde{\varphi}) = \varphi(f) \text{ for all } f \in L^\infty(G) \text{ and } \varphi \in P(G).$$

In particular, $E$ is a TIIM on $L^\infty(G)$; here we give another proof of this fact.

**Lemma 1.** Let $(e_i)$ be a bounded approximate identity of $L^1(G)$ and $e_i \to E (wk^*)$, then $E \in L^\infty(G)^*$ is a TIIM and $E(f) = f(e)$ for $f \in CB(G)$.

**Proof.** Let $\varphi \in P(G)$, $f \in L^\infty(G)$, with the aid of the relations $f.\varphi = (\frac{1}{\Delta} \tilde{\varphi}) \ast f$ and $\varphi.f = f \ast \varphi$ we have

$$E((\frac{1}{\Delta} \tilde{\varphi}) \ast f) = E(f.\varphi) = \lim_i e_i(f.\varphi) = \lim_i f.\varphi(e_i) = \lim_i f(\varphi \ast e_i) = f(\varphi)$$

and

$$E(f \ast \varphi) = E(\varphi.f) = \lim_i e_i(\varphi.f) = \lim_i \varphi.f(e_i) = \lim_i f(e_i \ast \varphi) = f(\varphi)$$

And hence $E((\frac{1}{\Delta} \tilde{\varphi} \ast f) = E(f \ast \varphi)$. By [6, Th.2] $E(f) = f(e)$ for $f \in CB(G)$.

For each locally compact group $G$, $L^1(G)$ has a bounded approximate identity, therefore by the above lemma and theorem of Banach Alaoglu, $L^\infty(G)$ always has a topological inner invariant mean.

**Theorem 2.** Let $G$ be a locally compact group. Then each TIIM on $LUC(G)$ has a topological inner invariant extension to $L^\infty(G)$.

**Proof.** Let $(e_i)$ be the bounded approximate identity of $L^1(G)$ in $P(G)$, and $m$ be a TIIM on $LUC(G)$. We choose an ultrafilter $\Gamma$ on the index set of $(e_i)$ that dominates the order filter and define $M : L^\infty(G) \to \mathbb{C}$ by $M(f) =$
\[ \lim_{\Gamma} m(\frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{e}_\gamma). \]  

\( M \) is a mean on \( L^\infty(G) \). Now for \( f \in L^\infty(G), \varphi \in P(G) \) and an arbitrary \( \epsilon > 0 \) there exists \( \gamma_0 \) such that if \( \gamma \geq \gamma_0 \) then

\[ | M(\frac{1}{\Delta} \tilde{\varphi} \ast f) - m(\frac{1}{\Delta} \tilde{e}_\gamma \ast \frac{1}{\Delta} \tilde{\varphi} \ast f \ast \tilde{e}_\gamma) | < \epsilon \]  \hspace{1cm} (1)

and

\[ | M(f \ast \tilde{\varphi}) - m(\frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{\varphi} \ast \tilde{e}_\gamma) | < \epsilon \]  \hspace{1cm} (2)

also we have

\[ \lim m(\frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{\varphi}) = \lim m(\frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{e}_\gamma \ast \tilde{\varphi}) = \lim m(\frac{1}{\Delta} \tilde{\varphi} \ast \frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{e}_\gamma) \]

\[ = \lim m(\frac{1}{\Delta} \tilde{\varphi} \ast f \ast \tilde{e}_\gamma) \]  \hspace{1cm} (3)

Now with (1), (2), (3) and some calculations we have

\[ | M(\frac{1}{\Delta} \tilde{\varphi} \ast f) - M(f \ast \tilde{\varphi}) | < \epsilon \]

Hence \( M(\frac{1}{\Delta} \tilde{\varphi} \ast f) = M(f \ast \tilde{\varphi}) \) and so \( M \) is a TIIM. Also for \( f \in LUC(G) \) we have \( \frac{1}{\Delta} \tilde{e}_\gamma \ast f \ast \tilde{e}_\gamma \rightarrow f \) in \( \| \cdot \|_u \), Therefore \( M \) is an extention of \( m \). \hspace{1cm} \( \blacksquare \)

Now we give a simpler proof based on the technic used in [7, Th.3.2] for the following well known theorem, see [9].

**Theorem 3.** Let \( G \) be a locally compact group. If \( G \) is amenable, then \( G \) is topological amenable.

**Proof.** By amenability of \( G \) there exists a net \( \{ \psi_\alpha \} \subset P(G) \) such that

\[ \lim_\alpha ||y(\psi_\alpha) - \psi_\alpha||_1 = 0 \]  uniformly on compacta; see [9]. Let \( \varphi \in P(G) \) with \( K = Supp(\varphi) \) compact. For every \( \epsilon > 0 \) we can find \( \alpha_0 \) such that if \( \alpha \geq \alpha_0 \), then \( ||y(\psi_\alpha) - \psi_\alpha||_1 < \epsilon \) for all \( \gamma \in K^{-1} \). Hence we have

\[ ||\varphi \ast \psi_\alpha - \psi_\alpha||_1 = \int_G || \int_G \Delta(y^{-1}) \varphi(y^{-1}) \psi_\alpha(yx)dy - \int_G \Delta(y^{-1}) \varphi(y^{-1}) \psi_\alpha(x)dy || dx \]

\[ \leq \int_G ||y(\psi_\alpha) - \psi_\alpha||_1 \Delta(y^{-1}) \varphi(y^{-1})dy \]

\[ = \int_{K^{-1}} ||y(\psi_\alpha) - \psi_\alpha||_1 \Delta(y^{-1}) \varphi(y^{-1})dy < \epsilon \]

Therefore \( \lim_\alpha ||\varphi \ast \psi_\alpha - \psi_\alpha||_1 = 0 \) and hence we have \( \lim_\alpha ||\varphi \ast \psi_\alpha - \psi_\alpha||_1 = 0 \) for all \( \varphi \in P(G) \), that is \( G \) is topological amenable. \hspace{1cm} \( \blacksquare \)
References


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