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Extention of Topological Inner Invariant Means

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Abstract

For a locally compact group G, we prove that a topological inner invariant mean on $LUC(G)$ has an extension to a topological inner invariant mean on $L^\infty(G)$.

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1 Introduction

Let G be a locally compact group with identity e and left Haar measure dx. Let $L^\infty(G)$ and $L^1(G)$ denote the spaces of essentially bounded functions and integrable functions respectively.

For a function $f : G \to \mathbb{C}$ we put ${}_xf(y) = f(xy)$ and $f_x(y) = f(yx)$ for all $x, y \in G$. Recall that a mean m on a linear subspace X of $L^\infty(G)$ containing the constants is a linear functional such that $||m|| = m(1) = 1$. If ${}_xf, f_x \in X$ for all $f \in X$ and $x \in G$, we say that m is inner invariant if $m({}_xf) = m(f_x)$ for all $x \in G$ and $f \in X$.

We denote by $P(G)$ the set of all $\varphi \in L^1(G)$ with $\varphi \geq 0$ and $||\varphi||_1 = 1$. If X is a linear subspace of $L^\infty(G)$ containing the constants, X is said to be topological left (resp. right) invariant if $P(G) * X \subset X$ (resp. $X * P(G)\tilde{} \subset X$). Here $\tilde{\varphi}$ is the function defined by $\tilde{\varphi}(x) = \varphi(x^{-1})$ for all $x \in G$ and $*$ denotes the convolution product of functions on G. Let X be topological invariant (i.e. topological left and right invariant). A mean defined on X is said to be topological inner invariant if

$$m(\frac{1}{\Delta}\tilde{\varphi} * f) = m(f * \tilde{\varphi}) \text{ for all } f \in X \text{ and } \varphi \in P(G)$$

where Δ is the modular function of G. The concept of topological inner invariant means (TIIM) was introduced and studied by Nasr-Isfahani [8] for a large class of Banach algebras containing $L^1(G)$ known as Lau algebras.
In [7] authors study locally compact groups for which $L^\infty(G)$ has a TIIM whose restriction to $C_b(G)$ is not δ_e. In this paper we show that a TIIM on $LUC(G)$ (space of left uniformly continuous functions on G) has a topological inner invariant extension to $L^\infty(G)$. Also based on the technic used in [7, Th.3.2] we give a simpler proof of the well known fact that amenability implies topological amenability (see [9]).

2 Topological inner invariant means

It is well known [7] that any extension E of δ_e from $C_b(G)$ to a mean on $L^\infty(G)$ is a mixed identity of $L^\infty(G)^*$; that is

$$E(\frac{1}{\Delta}\tilde{\varphi} * f) = E(f * \tilde{\varphi}) = \varphi(f) \text{ for all } f \in L^\infty(G) \text{ and } \varphi \in P(G).$$

In particular, E is a TIIM on $L^\infty(G)$; here we give another proof of this fact.

Lemma 1. *Let (e_i) be a bounded approximate identity of $L^1(G)$ and $e_i \to E$ (wk^*), then $E \in L^\infty(G)^*$ is a TIIM and $E(f) = f(e)$ for $f \in CB(G)$.*

Proof. Let $\varphi \in P(G)$, $f \in L^\infty(G)$, with the aid of the relations $f.\varphi = (\frac{1}{\Delta}\tilde{\varphi}) * f$ and $\varphi.f = f * \tilde{\varphi}$ we have

$$E((\frac{1}{\Delta}\tilde{\varphi}) * f) = E(f.\varphi) = \lim_i e_i(f.\varphi) = \lim_i f.\varphi(e_i) = \lim_i f(\varphi * e_i) = f(\varphi)$$

and

$$E(f * \tilde{\varphi}) = E(\varphi.f) = \lim_i e_i(\varphi.f) = \lim_i \varphi.f(e_i) = \lim_i f(e_i * \varphi) = f(\varphi)$$

And hence $E((\frac{1}{\Delta}\tilde{\varphi} * f) = E(f * \tilde{\varphi})$. By [6,Th.2] $E(f) = f(e)$ for $f \in CB(G)$. ■

For each locally compact group G, $L^1(G)$ has a bounded approximate identity, therefore by the above lemma and theorem of Banach Alaoghlu, $L^\infty(G)$ always has a topological inner invariant mean.

Theorem 2. *Let G be a locally compact group. Then each TIIM on $LUC(G)$ has a topological inner invariant extention to $L^\infty(G)$.*

Proof. Let (e_γ) be the bounded approximate identity of $L^1(G)$ in $P(G)$, and m be a TIIM on $LUC(G)$. We choose an ultrafilter Γ on the index set of (e_γ) that dominates the order filter and define $M : L^\infty(G) \to \mathbb{C}$ by $M(f) =$

$\lim_\Gamma m(\frac{1}{\Delta}\tilde{e_\gamma}*f*\tilde{e_\gamma})$. M is a mean on $L^\infty(G)$. Now for fix $f \in L^\infty(G)$, $\varphi \in P(G)$ and an arbitrary $\epsilon > 0$ there exists γ_0 such that if $\gamma \geq \gamma_0$ then

$$| M(\frac{1}{\Delta}\tilde{\varphi} * f) - m(\frac{1}{\Delta}\tilde{e_\gamma} * \frac{1}{\Delta}\tilde{\varphi} * f * \tilde{e_\gamma}) |< \epsilon \qquad (1)$$

and

$$| M(f * \tilde{\varphi}) - m(\frac{1}{\Delta}\tilde{e_\gamma} * f * \tilde{\varphi} * \tilde{e_\gamma}) |< \epsilon \qquad (2)$$

also we have

$$\lim m(\frac{1}{\Delta}\tilde{e_\gamma} * f * \tilde{\varphi}) = \lim m(\frac{1}{\Delta}\tilde{e_\gamma} * f * \tilde{e_\gamma} * \tilde{\varphi}) = \lim m(\frac{1}{\Delta}\tilde{\varphi} * \frac{1}{\Delta}\tilde{e_\gamma} * f * \tilde{e_\gamma})$$

$$= \lim m(\frac{1}{\Delta}\tilde{\varphi} * f * \tilde{e_\gamma}) \quad (3)$$

Now with (1), (2), (3) and some calculations we have

$$| M(\frac{1}{\Delta}\tilde{\varphi} * f) - M(f * \tilde{\varphi}) |< \epsilon$$

Hence $M(\frac{1}{\Delta}\tilde{\varphi} * f) = M(f * \tilde{\varphi})$ and so M is a TIIM. Also for $f \in LUC(G)$ we have $\frac{1}{\Delta}\tilde{e_\gamma} * f * \tilde{e_\gamma} \to f(in \parallel . \parallel_u)$, Therefore M is an extention of m. ■

Now we give a simpler proof based on the technic used in [7,Th.3.2] for the following well known theorem , see [9].

Theorem 3. *Let G be a locally compact group. If G is amenable, then G is topological amenable.*

Proof. By amenability of G there exists a net $\{\psi_\alpha\} \subset P(G)$ such that $\lim_\alpha ||_y(\psi_\alpha) - \psi_\alpha||_1 = 0$ uniformly on compacta; see [9]. Let $\varphi \in P(G)$ with $K = Supp(\varphi)$ compact. For every $\epsilon > 0$ we can find α_0 such that if $\alpha \geq \alpha_0$, then $||_y(\psi_\alpha) - \psi_\alpha||_1 < \epsilon$ for all $y \in K^{-1}$. Hence we have

$$||\varphi*\psi_\alpha - \psi_\alpha||_1 = \int_G | \int_G \Delta(y^{-1})\varphi(y^{-1})\psi_\alpha(yx)dy - \int_G \Delta(y^{-1})\varphi(y^{-1})\psi_\alpha(x)dy|dx$$

$$\leq \int_G ||_y(\psi_\alpha) - \psi_\alpha||_1 \Delta(y^{-1})\varphi(y^{-1})dy$$

$$= \int_{K^{-1}} ||_y(\psi_\alpha) - \psi_\alpha||_1 \Delta(y^{-1})\varphi(y^{-1})dy < \epsilon$$

Therefore $\lim_\alpha ||\varphi * \psi_\alpha - \psi_\alpha||_1 = 0$ and hence we have $\lim_\alpha ||\varphi * \psi_\alpha - \psi_\alpha||_1 = 0$ for all $\varphi \in P(G)$, that is G is topological amenable. ■

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