Subclasses of p-Valent and Prestarlike Functions

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Abstract

The object of the present paper is to investigate coefficients for functions belonging to the subclasses $R^p[\alpha, \beta]$ and $C^p[\alpha, \beta]$ of p-valent $\alpha$-prestarlike functions of order $\beta$ with negative coefficients. We obtain closure theorems, integral operators, radius of starlikeness and distortion theorems for functions belonging to the classes $R^p[\alpha, \beta]$ and $C^p[\alpha, \beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R^p[\alpha, \beta]$ and $C^p[\alpha, \beta]$.

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1 Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, \ldots\})$$

(1.1)

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order $\alpha(0 \leq \alpha < p)$ if $f(z)$ satisfies the conditions
\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U) \] (1.2)

and

\[ \int_{0}^{2\pi} \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta = 2\pi p \quad (z \in U). \] (1.3)

We denote by \( S^*(p, \alpha) \) the class of p-valent starlike functions of order \( \alpha \). The class \( S^*(p, \alpha) \) was introduced by Patil and Thakare [8].

The function

\[ s^p_\alpha(z) = \frac{z^p}{(1 - z)^{2(p - \alpha)}} \quad (0 \leq \alpha < p; p \in N) \] (1.4)

is the familiar extremal function for the class \( S^*(p, \alpha) \). Setting

\[ G^p(\alpha, n) = \frac{\prod_{m=2}^{n} [2(p - \alpha) + m - 2]}{(n - 1)!} \quad (n \in N/\{1\}, 0 \leq \alpha < p), \] (1.5)

\( s^p_\alpha(z) \) can be written in the form:

\[ s^p_\alpha(z) = z^p + \sum_{n=1}^{\infty} G^p(\alpha, n + 1) z^{p+n}. \] (1.6)

Clearly, \( s^p_\alpha(z) \in S^*(p, \alpha) \) and \( G^p(\alpha, n + 1) \) is a decreasing function in \( \alpha(0 \leq \alpha \leq \frac{2p-1}{2}; p \in N) \) and satisfies

\[ \lim_{n \to \infty} G^p(\alpha, n + 1) = \begin{cases} \infty & (\alpha < \frac{2p-1}{2}) \\ 1 & (\alpha = \frac{2p-1}{2}) \\ 0 & (\alpha > \frac{2p-1}{2}) \end{cases}. \]

Let \((f * g)(z)\) denote the Hadamard product (or convolution) of the functions \( f(z) \) and \( g(z) \), that is, if \( f(z) \) is given by (1.1) and \( g(z) \) is given by

\[ g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \] (1.7)

then
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\[(f \ast g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}. \quad (1.8)\]

A function \(f(z) \in A(p)\) is said to be \(p\)-valent \(\alpha\)-prestarlike function of order \(\beta\) \((0 \leq \alpha < p; 0 \leq \beta < p; p \in N)\) if

\[(f \ast s_\alpha)(z) \in S^*(p, \beta), \quad (1.9)\]

where \(s_\alpha(z)\) is defined by (1.4). We denote by \(R^p(\alpha, \beta)\) the class of all \(p\)-valent \(\alpha\)-prestarlike functions of order \(\beta\). For \(\alpha = \frac{2p-1}{2}; 0 \leq \beta < p; p \in N, R^p(\frac{2p-1}{2}, \beta) = S^*(p, \beta)\). Further let \(C^p(\alpha, \beta)\) be the subclass of \(A(p)\) consisting of functions \(f(z)\) satisfying

\[f(z) \in C^p(\alpha, \beta) \text{ if and only if } \frac{zf'(z)}{p} \in R^p(\alpha, \beta). \quad (1.10)\]

We note that:

(i) \(R^p(\alpha, \alpha) = R^p(\alpha)(0 \leq \alpha < p; p \in N)\), the class of \(p\)-valent prestarlike functions of order \(\alpha\), was studied by Kumar and Reddy [3] and Shenen et al. [13];

(ii) \(R^1(\alpha, \beta) = R_{\alpha, \beta}(0 \leq \alpha < 1; 0 \leq \beta < 1)\), the class of \(\alpha\)-prestarlike functions of order \(\beta\), was introduced by Sheil-Small et al. [12];

(iii) \(R^1(\alpha, \alpha) = R_{\alpha}(0 \leq \alpha < 1)\), the class of prestarlike functions of order \(\alpha\), was introduced by Ruscheweyh [10];

(iv) \(C^1(\alpha, \beta) = C(\alpha, \beta)(0 \leq \alpha < 1; 0 \leq \beta < 1)\), the subclass of \(A(1) = A\) consisting of functions \(f(z) \in A\) satisfying \(zf'(z) \in R^1(\alpha, \beta) = R(\alpha, \beta)\), was introduced by Owa and Uralegaddi [7].

Denoting by \(T(p)\) the subclass of \(A(p)\) consisting of functions of the form:

\[f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}(a_{p+n} \geq 0; p \in N). \quad (1.11)\]

We denote by \(S^*[p, \beta]\), \(R^p[\alpha, \beta]\) and \(C^p[\alpha, \beta]\) the classes obtained by taking intersections, respectively, of the classes \(S^*(p, \beta)\), \(R^p(\alpha, \beta)\) and \(C^p(\alpha, \beta)\) with the class \(T(p)\). Thus, we have

\[S^*[p, \beta] = S^*(p, \beta) \cap T(p), \quad (1.12)\]

\[R^p[\alpha, \beta] = R^p(\alpha, \beta) \cap T(p), \quad (1.13)\]

and

\[C^p[\alpha, \beta] = C^p(\alpha, \beta) \cap T(p). \quad (1.14)\]
The class $S^{*}[p, \beta]$ was studied by Owa [5]. It follows from (1.13) and (1.14) that

$$f(z) \in C^{p}[\alpha, \beta] \text{ if and only if } \frac{zf'(z)}{p} \in R^{p}[\alpha, \beta].$$  \hspace{1cm} (1.15)

Also we note that, by specializing the parameters $\alpha, \beta$ and $p$, we obtain the following subclasses studied by various authors:

(i) $R^{p}[\alpha, \alpha] = R^{p}[\alpha](0 \leq \alpha < 1; p \in N)$ (Kumar and Reddy [3]);

(ii) $R^{p}[\alpha, \alpha] = R[\alpha](0 \leq \alpha < 1)$ (Silverman and Silvia [14] and Owa and Al-Bassam [6]);

(iii) $R^{p}[\alpha, \beta] = R[\alpha, \beta](0 \leq \alpha < 1, 0 \leq \beta < 1)$ (Silverman and Silvia [15], Uralegaddi and Sarangi [17], Aouf and Salagean [2], Aouf et al. [1], Srivastava and Aouf [16] and Rania and Srivastava [9]).

In the present paper we investigate coefficients for functions belonging to the subclasses $R^{p}[\alpha, \beta]$ and $C^{p}[\alpha, \beta]$ of $p$-valent $\alpha$-prestarlike functions of order $\beta$ with negative coefficients. We obtain closure theorems, integral operators, radii of starlikeness and convexity and distortion theorems for functions belonging to the classes $R^{p}[\alpha, \beta]$ and $C^{p}[\alpha, \beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R^{p}[\alpha, \beta]$ and $C^{p}[\alpha, \beta]$.

2 Coefficient inequalities

**Theorem 1.** Let the function $f(z)$ be defined by (1.11). Then $f(z)$ is in the class $R^{p}[\alpha, \beta]$ if and only if

$$\sum_{n=1}^{\infty} (n + p - \beta)G^{p}(\alpha, n + 1)a_{p+n} \leq (p - \beta).$$  \hspace{1cm} (2.1)

**Proof.** It is known [5] that a necessary and sufficient condition for $g(z) = z^{p} - \sum_{n=1}^{\infty} b_{p+n}z^{p+n} (b_{p+n} \geq 0)$ to be in the class $S^{*}[p, \beta]$ is that

$$\sum_{n=1}^{\infty} (n + p - \beta)b_{p+n} \leq (p - \beta).$$  \hspace{1cm} (2.2)

Since

$$(f \ast s^{p}_{\alpha})(z) = z^{p} - \sum_{n=1}^{\infty} G^{p}(\alpha, n + 1)a_{p+n}z^{p+n},$$  \hspace{1cm} (2.3)

where $s^{p}_{\alpha}(z)$ is given by (1.4), the result follows.
Corollary 2. If \( f(z) \) is in the class \( R^p[\alpha, \beta] \), then

\[
a_{p+n} \leq \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} \quad (p, n \in \mathbb{N}),
\]

with equality for

\[
f(z) = z^p - \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)}z^{p+n} \quad (p, n \in \mathbb{N}).
\]

In view of (1.15), Theorem 1 yields the following necessary and sufficient condition for the class \( C^p[\alpha, \beta] \).

Theorem 3. The function \( f(z) \), defined by (1.11), is in the class \( C^p[\alpha, \beta] \) if and only if

\[
\sum_{n=1}^{\infty} \frac{(p+n)}{p} (n + p - \beta)G^p(\alpha, n + 1)a_{p+n} \leq (p - \beta).
\]

3 Extreme points

From Theorem 1 and Theorem 2, we see that both \( R^p[\alpha, \beta] \) and \( C^p[\alpha, \beta] \) are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

Theorem 4. Let

\[
f_p(z) = z^p
\]

and

\[
f_{p+n}(z) = z^p - \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)}z^{p+n} \quad (p, n \in \mathbb{N}).
\]

Then \( f(z) \in R^p[\alpha, \beta] \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=0}^{\infty} \mu_{p+n}f_{p+n}(z),
\]

where \( \mu_{p+n} \geq 0 \) and \( \sum_{n=0}^{\infty} \mu_{p+n} = 1 \).

Proof. Suppose that
\[ f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z) \]
\[ = z^p - \sum_{n=1}^{\infty} \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} \mu_{p+n} z^{p+n}. \quad (3.4) \]

Then it follows that
\[
\sum_{n=1}^{\infty} \frac{(n + p - \beta)G^p(\alpha, n + 1)}{(p - \beta)} \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} \mu_{p+n} \]
\[ = \sum_{n=1}^{\infty} \mu_{p+n} = 1 - \mu_p \leq 1. \quad (3.5) \]

Therefore, by Theorem 1, \( f(z) \in R^p[\alpha, \beta] \).

Conversely, assume that the function \( f(z) \) defined by (1.11) belongs to the class \( R^p[\alpha, \beta] \). Then
\[ a_{p+n} \leq \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} (p, n \in \mathbb{N}). \quad (3.6) \]

Setting
\[ \mu_{p+n} = \frac{(n + p - \beta)G^p(\alpha, n + 1)}{(p - \beta)} (p, n \in \mathbb{N}) \quad (3.7) \]
and
\[ \mu_p = 1 - \sum_{n=1}^{\infty} \mu_{p+n}, \quad (3.8) \]
we see that \( f(z) \) can be expressed in the form (3.3). This completes the proof of Theorem 3.

**Corollary 5**. The extreme points of the class \( R^p[\alpha, \beta] \) are the functions \( f_p(z) = z^p \) and
\[
f_{p+n}(z) = z^p - \frac{(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} z^{p+n} (p, n \in \mathbb{N}). \]

Similarly, we have

**Theorem 6**. Let
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\[ f_p(z) = z^p \]  \hspace{1cm} (3.9)

and

\[ f_{p+n}(z) = z^p - \frac{(p - \beta)}{(\frac{p}{p})(n + p - \beta)G^p(\alpha, n + 1)}z^{p+n} (p, n \in N). \]  \hspace{1cm} (3.10)

Then \( f(z) \in C^p[\alpha, \beta] \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=0}^{\infty} \mu_{p+n}f_{p+n}(z), \]  \hspace{1cm} (3.11)

where \( \mu_{p+n} \geq 0 \) and \( \sum_{n=0}^{\infty} \mu_{p+n} = 1. \)

**Corollary 7.** The extreme points of the class \( C^p[\alpha, \beta] \) are the functions \( f_p(z) = z^p \) and

\[ f_{p+n}(z) = z^p - \frac{(p - \beta)}{(\frac{p}{p})(n + p - \beta)G^p(\alpha, n + 1)}z^{p+n} (p, n \in N). \]

### 4 Distortion theorems

In view of Theorems 3 and 4, we will obtain distortion theorems for the classes \( R^p[\alpha, \beta] \) and \( C^p[\alpha, \beta]. \)

**Lemma 8.** For \( 0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p \) and \( p \in N, \) then \( (n + p - \beta)G^p(\alpha, n + 1) \) is an increasing function of \( n, \) where \( G^p(\alpha, n + 1) \) is defined by (1.5).

**Proof.** Let \( K(\alpha, \beta, n, p) = (n + p - \beta)G^p(\alpha, n + 1). \) Since,

\[ G^p(\alpha, n + 2) = \frac{2p + n - 2\alpha}{n + 1} G^p(\alpha, n + 1), \]  \hspace{1cm} (4.1)

we can see that \( K(\alpha, \beta, n + 1, p) \geq K(\alpha, \beta, n, p) \) if and only if

\[ 2(p - \alpha)(n + 1 + p - \beta) - (p - \beta) \geq 0, \]  \hspace{1cm} (4.2)

for \( 0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p \) which holds for \( p \in N. \) This completes the proof of Lemma 1.

In the remainder of this section, we assume that \( f(z) \) is defined by (1.11), \( 0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p \) and \( p \in N. \)
Theorem 9. If $f(z)$ is in the class $R^p[\alpha, \beta]$, then

$$|z|^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^{p+1}$$

$(z \in U)$.

Equality holds for the function $f_{p+1}(z)$ given by

$$f_{p+1}(z) = z^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}z^{p+1} (z \in U).$$

Proof. By virtue of Theorem 3, we note that

$$|z|^p - \max_{n \in \mathbb{N}} \frac{(p-\beta)}{(n+p-\beta)G_p(\alpha,n+1)}|z|^{p+n} \leq |f(z)| \leq |z|^p + \max_{n \in \mathbb{N}} \frac{(p-\beta)}{(n+p-\beta)G_p(\alpha,n+1)}|z|^{p+n} .$$

(4.5)

From Lemma 1, we see that the max in (4.5) occurs when $n = 1$. This completes the proof of Theorem 5.

Theorem 10. If $f(z)$ is in the class $R^p[\alpha, \beta]$, then

$$p|z|^{p-1} - \frac{(p+1)(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{(p+1)(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^p$$

$(z \in U)$.

Equality holds for $f_{p+1}(z)$ given by (4.4).

Proof. We know that

$$p|z|^{p-1} - \max_{n \in \mathbb{N}} \frac{(p-\beta)(n+p)|z|^{n+p-1}}{(n+p-\beta)G_p(\alpha,n+1)} \leq |f'(z)| \leq p|z|^{p-1} + \max_{n \in \mathbb{N}} \frac{(p-\beta)(n+p)|z|^{n+p-1}}{(n+p-\beta)G_p(\alpha,n+1)} (z \in U).$$

(4.7)

From Lemma 1, we see that the max in (4.7) occurs when $n = 1$. This completes the proof of Theorem 6.

Theorem 11. If $f(z)$ is in the class $C^p[\alpha, \beta]$, then
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\[ |f(z)| \geq |z|^p - \frac{(p-\beta)}{2\left(\frac{1}{p} + p - \beta\right)(p-\alpha)}|z|^{p+1} \quad (4.8) \]

and

\[ |f(z)| \leq |z|^p + \frac{(p-\beta)}{2\left(\frac{1}{p} + p - \beta\right)(p-\alpha)}|z|^{p+1} \quad (4.9) \]

for \( z \in U \). The results are sharp for the function \( f(z) \) given by

\[ f(z) = z^p - \frac{(p-\beta)}{2\left(\frac{1}{p} + p - \beta\right)(p-\alpha)}z^{p+1} \quad (z \in U). \quad (4.10) \]

**Proof.** From Theorem 4, we have that

\[ |f(z)| \geq |z|^p - \max_{n \in \mathbb{N}} \frac{(p-\beta)}{\left(\frac{1}{p} + n - \beta\right)G_p(\alpha, n+1)} |z|^{p+n} \quad (4.11) \]

and

\[ |f(z)| \leq |z|^p + \max_{n \in \mathbb{N}} \frac{(p-\beta)}{\left(\frac{1}{p} + n - \beta\right)G_p(\alpha, n+1)} |z|^{p+n} \quad (4.12) \]

for \( z \in U \). From Lemma 1, we see that the max in (4.11) and (4.12) occur when \( n = 1 \). This completes the proof of Theorem 7.

**Corollary 12.** If \( f(z) \) is in the class \( C^p[\alpha, \beta] \). Then \( f(z) \) is included in a disc with its center at the origin and radius \( r \) given by

\[ r = 1 + \frac{(p-\beta)}{2\left(\frac{1}{p} + p - \beta\right)(p-\alpha)} \cdot (4.13) \]

**Theorem 13.** If \( f(z) \) is in the class \( C^p[\alpha, \beta] \), then

\[ |f'(z)| \geq p|z|^{p-1} - \frac{p(p-\beta)}{2\left(1 + p - \beta\right)(p-\alpha)}|z|^p \quad (4.14) \]

and

\[ |f'(z)| \leq p|z|^{p-1} + \frac{p(p-\beta)}{2\left(1 + p - \beta\right)(p-\alpha)}|z|^p \quad (4.15) \]

for \( z \in U \). The bounds for (4.14) and (4.15) are sharp for the function \( f(z) \) given by (4.10).
Proof. By means of Theorem 4, we note that

\[
|f'(z)| \geq p |z|^{p-1} - \max_{n \in \mathbb{N}} \frac{p(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} |z|^{p+n-1}
\]  

(4.16)

and

\[
|f'(z)| \leq p |z|^{p-1} + \max_{n \in \mathbb{N}} \frac{p(p - \beta)}{(n + p - \beta)G^p(\alpha, n + 1)} |z|^{p+n-1}.
\]  

(4.17)

Also by using Lemma 1, we see that the max in (4.16) and (4.17) occur when \(n = 1\). This completes the proof of Theorem 8.

Remark 1. Making use of the relationship (1.15) between the classes \(R^p[\alpha, \beta]\) and \(C^p[\alpha, \beta]\), we can deduce Theorem 8 from Theorem 5.

5 Integral operators

Theorem 14. Let the function \(f(z)\) defined by (1.11) be in the class \(R^p[\alpha, \beta]\), and let \(c\) be a real number such that \(c > -p\). Then the function \(F(z)\) defined by

\[
F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) \, dt
\]  

(5.1)

also belongs to the class \(R^p[\alpha, \beta]\).

Proof. From the representation of \(F(z)\), it follows that

\[
F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},
\]  

(5.2)

where

\[
b_{p+n} = \left(\frac{c+p}{c+p+n}\right)a_{p+n}.
\]

Therefore

\[
\sum_{n=1}^{\infty} (n + p - \beta)G^p(\alpha, n + 1)b_{p+n} = \sum_{n=1}^{\infty} (n + p - \beta)G^p(\alpha, n + 1)(\frac{c+p}{c+p+n})a_{p+n}
\]

\[
\leq \sum_{n=1}^{\infty} (n + p - \beta)G^p(\alpha, n + 1)a_{p+n} \leq (p - \beta),
\]

since \(f(z) \in R^p[\alpha, \beta]\). Hence, by Theorem 1, \(F(z) \in R^p[\alpha, \beta]\).
Corollary 15. Under the same conditions as Theorem 9, a similar proof shows that the function $F(z)$ defined by (5.1) is in the class $C^p[\alpha, \beta]$, whenever $f(z)$ is in the class $C^p[\alpha, \beta]$.

6 Radius of starlikeness

Since $f(z)$ defined by (1.11) is $p$-valent in the unit disc $U$ if

$$
\sum_{n=1}^{\infty} \frac{p+n}{p} a_{p+n} \leq 1 \quad (cf. [4])
$$

we can see that $R^p[\alpha, \beta]$ is a subclass of $T(p)$ if $0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p, p \in N$ with the aid of Theorem 1. Also we can see that $C^p[\alpha, \beta]$ is a subclass of $T(p)$ if $0 \leq \alpha \leq \frac{(2p-1)(p-\beta) + 2p}{2(1+p-\beta)}, 0 \leq \beta < p, p \in N$ with the aid of Theorem 2.

Theorem 16. Let the function $f(z)$ defined by (1.11) be in the class $R^p[\alpha, \beta]$.

Then $f(z)$ is $p$-valently starlike of order $\delta (0 \leq \delta < p)$ in $|z| < r_1$, where

$$
r_1 = \inf_{n \in N} \left\{ \frac{(p-\delta)(n+p-\beta)G^p(\alpha, n+1)}{(n+p-\beta)(p-\beta)} \right\} \frac{1}{n} \quad (n \geq 1).
$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{for} \quad |z| < r_1.
$$

We have

$$
\left| \frac{zf'(z)}{f(z)} - p \right| \leq \sum_{n=1}^{\infty} \frac{n a_{p+n} |z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n} |z|^n}.
$$

Thus

$$
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{if}
$$

$$
\sum_{n=1}^{\infty} \frac{(n+p-\delta)}{(p-\delta)} a_{p+n} |z|^n \leq 1.
$$

(6.3)

Hence, by Theorem 1, (6.3) will be true if

$$
\frac{(n+p-\delta)}{(p-\delta)} |z|^n \leq \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)}
$$
or if

\[
|z| \leq \left\{ \frac{(p - \delta)(n + p - \beta)G^p(\alpha, n + 1)}{(n + p - \delta)(p - \beta)} \right\} \frac{1}{n} \quad (n \geq 1). \tag{6.4}
\]

The theorem follows easily from (6.4).

**Corollary 17.** Let the function \( f(z) \) defined by (1.11) be in the class \( R^p[\alpha, \beta] \). Then \( f(z) \) is \( p \)-valently convex of order \( \delta \) \((0 \leq \delta < p)\) in \(|z| < r_2\), where

\[
r_2 = \inf_{n} \left\{ \frac{p(p - \delta)(n + p - \beta)G^p(\alpha, n + 1)}{(n + p)(n + p - \delta)(p - \beta)} \right\} \frac{1}{n} \quad (n \geq 1). \tag{6.5}
\]

The result is sharp, with the extremal function \( f(z) \) given by (2.5).

**Theorem 18.** Let the function \( f(z) \) defined by (1.11) be in the class \( C^p[\alpha, \beta] \), \( 0 \leq \alpha \leq \frac{(2p - 1)(p - \beta) + 2p}{2(1 + p - \beta)} \), \( 0 \leq \beta < p \) and \( p \in N \). Then \( f(z) \) is \( p \)-valently starlike of order \( \delta \) \((0 \leq \delta < p)\) in \(|z| < r_3\), where

\[
r_3 = \inf_{n \in N} \left\{ \frac{(p - \delta)(p + n)(n + p - \beta)G^p(\alpha, n + 1)}{p(p - \beta)(n + p - \delta)} \right\} \frac{1}{n}. \tag{6.6}
\]

The result is sharp, with the extremal function \( f(z) \) given by

\[
f(z) = z^p - \frac{(p - \beta)}{(p + n)(n + p - \beta)G^p(\alpha, n + 1)} z^{p + n} \quad (p, n \in N). \tag{6.7}
\]

**Corollary 19.** Let the function \( f(z) \) defined by (1.11) be in the class \( C^p[\alpha, \beta] \), \( 0 \leq \alpha \leq \frac{(2p - 1)(p - \beta) + 2p}{2(1 + p - \beta)} \), \( 0 \leq \beta < p \) and \( p \in N \). Then \( f(z) \) is \( p \)-valently convex of order \( \delta \) \((0 \leq \delta < p)\) in \(|z| < r_4\), where

\[
r_4 = \inf_{n \in N} \left\{ \frac{(p - \delta)(n + p - \beta)G^p(\alpha, n + 1)}{(p - \beta)(n + p - \delta)} \right\} \frac{1}{n} \quad (n \geq 1). \tag{6.8}
\]

The result is sharp, with the extremal function \( f(z) \) given by (6.7).
7 Modified Hadamard products

Let the functions \( f_j(z) (j = 1, 2) \) be defined by (3.1). The modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) is defined by

\[
(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}.
\] (7.1)

**Theorem 20.** Let the functions \( f_j(z) (j = 1, 2) \) defined by (3.1) be in the class \( R^p[\alpha, \beta] \). Then \( (f_1 * f_2)(z) \in R^p[\alpha, \gamma(\alpha, \beta, p)] \), where

\[
\gamma(\alpha, \beta, p) = p - \frac{(p - \beta)^2}{2(1 + p - \beta)^2(p - \alpha) - (p - \beta)^2}.
\] (7.2)

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [11], we need to find the largest \( \gamma = \gamma(\alpha, \beta, p) \) such that

\[
\sum_{n=1}^{\infty} \frac{(n+p-\gamma)G^p(\alpha, n+1)}{(p-\gamma)} a_{p+n,1} a_{p+n,2} \leq 1.
\] (7.3)

Since

\[
\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,1} \leq 1
\] (7.4)

and

\[
\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,2} \leq 1,
\] (7.5)

by the Cauchy-Schwarz inequality we have

\[
\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1} a_{p+n,2}} \leq 1.
\] (7.6)

Thus it is sufficient to show that

\[
\frac{(n+p-\gamma)G^p(\alpha, n+1)}{(p-\gamma)} a_{p+n,1} a_{p+n,2} \leq \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1} a_{p+n,2}}
\] (n \geq 1),

that is, that...
\[ \sqrt{a_{p+n,1} a_{p+n,2}} \leq \frac{(n+p-\beta)(p-\gamma)}{(n+p-\gamma)(p-\beta)} \] (7.8)

Note that
\[ \sqrt{a_{p+n,1} a_{p+n,2}} \leq \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha, n+1)} (n \geq 1). \] (7.9)

Consequently, we need only to prove that
\[ \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha, n+1)} \leq \frac{(n+p-\beta)(p-\gamma)}{(n+p-\gamma)(p-\beta)} (n \geq 1) \] (7.10)

or, equivalently, that
\[ \gamma \leq p - \frac{n(p-\beta)^2}{(n+p-\beta)^2G^p(\alpha, n+1)- (p-\beta)^2} (n \geq 1). \] (7.11)

Since
\[ A(n) = p - \frac{n(p-\beta)^2}{(n+p-\beta)^2G^p(\alpha, n+1)- (p-\beta)^2} \] (7.12)

is an increasing function of \( n(n \geq 1) \) for \( 0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p \) and \( p \in \mathbb{N} \), letting \( n = 1 \) in (7.12), we obtain
\[ \gamma \leq A(1) = p - \frac{(p-\beta)^2}{2(1+p-\beta)^2(p-\alpha) - (p-\beta)^2}, \] (7.13)

which completes the proof of Theorem 12.

Finally, by taking the functions
\[ f_j(z) = z^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)} z^{p+1} (j = 1, 2) \] (7.14)

we can see that the result is sharp.

**Corollary 21.** For \( f_j(z)(j = 1, 2) \) as in Theorem 12, we have
\[ h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1} a_{p+n,2}} z^{p+n} \] (7.15)

belongs to the class \( R^p[\alpha, \beta] \).

The result follows from the inequality (7.6). It is sharp for the same functions as in Theorem 12.
Corollary 22. Let the functions $f_j(z)(j = 1, 2)$ defined by (3.1) be in the class $C^p[\alpha, \beta]$. Then $(f_1 \ast f_2)(z) \in C^p[\alpha, \lambda(\alpha, \beta, p)]$, where

$$
\lambda(\alpha, \beta, p) = p - \frac{(p - \beta)^2}{2\left(\frac{p+1}{p}\right)(1 + p - \beta)^2(p - \alpha) - (p - \beta)^2}.
$$

(7.16)

The result is sharp for the functions

$$
f_j(z) = z^p - \frac{(p - \beta)}{2\left(\frac{p+1}{p}\right)(1 + p - \beta)(p - \alpha)} z^{p+1} (j = 1, 2).
$$

(7.17)

References


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