Normalized Matching Property of a Class of Posets Consisting of Subspaces

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Abstract

Let $V_n(q)$ be the n -dimensional vector space over the finite field with q elements, and let $T_1, T_2, \dots, T_{r+1}$ be $r+1$ subspaces of $V_n(q)$ such that $V_n(q) = \bigoplus_{i=1}^{r+1} T_i$. Assuming that $\dim(T_i) = 2$ for $1 \leq i \leq r$, let $\mathcal{F}$ be the collection of all subspaces S 's of $V_n(q)$ such that $S = \bigoplus_{i=1}^{r+1} (S \cap T_i)$ and S contains at least one T_i for some $1 \leq i \leq r$. In this paper we will prove that $\mathcal{F}$ is log concave and has the normalized matching property.

Keywords: Poset, Subspace lattice, Sperner property, Normalized matching property

Mathematics Subject Classification: 05D05, 06A07

1 Introduction

A finite poset P is *graded* if every maximal chain of P has the same length, denoted by $r(P)$ which is called the rank of P . Thus, for each $x \in P$ every maximal chain with x as the top element has the same length, denoted by $r(x)$. Here the length of a chain with k elements is $k-1$. Let P_i denote the i th rank of P which consists of all $x \in P$ with $r(x) = i$, $i = 0, 1, 2, \dots, r(P)$. A graded poset P with rank n is *rank symmetric* if $P_i = P_{n-i}$ for $0 \leq i \leq n/2$. It is *rank unimodal* if $|P_0| \leq |P_1| \leq \dots \leq |P_k| \geq |P_{k+1}| \geq \dots \geq |P_n|$ for some $0 \leq k \leq n$. Similarly, we say P is *log concave* if $P_i^2 \geq P_{i-1}P_{i+1}$ for $1 \leq i \leq n-1$.

An *antichain* in P is a subset of P such that any two elements of this subset are incomparable. Clearly, each rank P_i of P is an antichain. The poset P is said to have *Sperner property* if the maximum size of an antichain equals the maximum of the Whitney numbers $|P_n|$. For $A \subseteq P_i$, define $\nabla(A) = \{b \in P_{i+1} : b \geq a \text{ for some } a \in A\}$. We say P has the *normalized matching property*

(NM), or is normal if

$$\frac{|\nabla(A)|}{|A|} \geq \frac{|P_{i+1}|}{|P_i|} \quad \text{for all } A \subseteq P_i, \quad i = 0, 1, 2, \dots, r(P) - 1. \quad (1)$$

We say P has the *LYM property*, or is a LYM poset if for every antichain A in P , the following inequality holds:

$$\sum_{i=0}^{r(P)} \frac{|A \cap P_i|}{|P_i|} \leq 1.$$

It is well known that the normalized matching property is equivalent to LYM property [9]. From [4] we know the following result.

Theorem 1.1 *Suppose that posets P and Q are normal and log concave. Then $P \times Q$ is also normal and log concave.*

Let $C(n, k)$ be the collection of all subsets of an n -set Y which intersect a fixed k -subset X . Then $C(n, k)$ is a natural generalization for the subset lattice. Lih [10] first observed this and showed that $C(n, k)$ has the Sperner property. Griggs [2] further showed that $C(n, k)$ has several strong properties. Let $X_1, X_2, \dots, X_r$ be r 2-subsets of Y , and H be the collections of all subsets of Y which contains at least one X_i . In [3] Horrocks showed that H is log concave and has the LYM property. In [11] Wang investigated the q -analogue of $C(n, k)$ and proved that $C[n, k]$ is sperner and log concave. In this paper we will investigate the q -analogue of H .

Let $V_n(q)$ denote an n -dimensional vector space over the finite field with q elements and $L(V_n(q))$ the lattice of subspaces of $V_n(q)$, ordered by inclusion. If there is no confusion, write them as V and $L(V)$ for short. It is well known that $L(V)$ is log concave and has the normalized matching property. Let $T_1, T_2, \dots, T_{r+1}$ be $r+1$ subspaces of V such that $V = \bigoplus_{i=1}^{r+1} T_i$. Assuming that $\dim(T_i) = 2$ for $1 \leq i \leq r$, let $\mathcal{F}$ be the collection of all subspaces S 's of V such that $S = \bigoplus_{i=1}^{r+1} (S \cap T_i)$ and S contains at least one T_i for some $1 \leq i \leq r$, and ordered by inclusion $\mathcal{F}$ is a graded poset. When $T_{r+1} = \{0\}$, we write $\mathcal{F}$ as $\mathcal{F}'$.

The main result of this paper is the following theorem.

Theorem 1.2 *$\mathcal{F}$ is log concave and has the normalized matching property.*

Note that if $T_{r+1} \neq \{0\}$, then $\mathcal{F}$ is the product of the two posets $\mathcal{F}'$ and $L(T_{r+1})$. By Theorem 1.1, to prove the main theorem we only need to prove that $\mathcal{F}'$ is log concave and has the normalized matching property. In section 2 we will prove that $\mathcal{F}'$ has the normalized matching property and in section 3 we will prove that $\mathcal{F}'$ is log concave. In the following, we always assume that $V = \bigoplus_{i=1}^r T_i$.

2 Normalized matching property of $\mathcal{F}'$

As stated in [7], some extremal problems can be considered in a weighted poset (P, w) , which is a poset P together with a function (called a weighted function) w from P into the set of non-negative real numbers. The weight $w(A)$ of a subset A of P is defined by $w(A) = \sum_{a \in A} w(a)$. Every poset P can be considered as a weighted poset (P, w) , where $w \equiv 1$, that is, $w(x) = 1$ for all $x \in P$. We say (P, w) has the Sperner property if some P_i is a maximal weighted antichain of P . We say (P, w) has the NM property if

$$\frac{w(\nabla(A))}{w(A)} \geq \frac{w(P_{i+1})}{w(P_i)} \text{ for all } A \subseteq P_i, i = 0, 1, 2, \dots, r(P) - 1. \quad (2)$$

Let Γ be a permutation acting on (P, w) which preserves the order relation and the weight on P , that is, for every $\gamma \in \Gamma$, $x \leq y$ in P implies $\gamma(x) \leq \gamma(y)$, and $w(\gamma(x)) = w(x)$. Then we have the quotient poset $(P/\Gamma, w_\Gamma)$, where P/Γ consists of the Γ -orbits ordered as follows: $\Gamma(x) \leq \Gamma(y)$ in P/Γ if $x' \leq y'$ for some $x' \in \Gamma(x)$ and $y' \in \Gamma(y)$, and the weight function w_Γ is given by $w_\Gamma(\Gamma x) = |\Gamma x|w(x)$. By a theorem due to Kleitman, Edelberg and Lubell [8] (see also [1]) we know that (P, w) has the Sperner property if and only if $(P/\Gamma, w_\Gamma)$ does. The NM version of this result was given as follows.

Theorem 2.1 [6] *(P, w) has the NM property if and only if $(P/\Gamma, w_\Gamma)$ does.*

From these results we see that when study the Sperner property or NM property of a poset, we need only consider its quotient poset, which is generally much smaller than itself. By this approach Wang respectively proved in [5] and [6] that the subgroup lattice $L_{k^n}(p)$ of an abelian p -group has the Sperner property and has the NM property for sufficiently large prime number p .

Let $V = \bigoplus_{i=1}^r T_i$, and $GL(V)$ denote the general linear group over the $2r$ -dimensional space V consisting of all invertible linear transformations of V . Let H be the subgroup of $GL(V)$ such that for any $f \in H$, $f(T_i) = T_j$ for $1 \leq i, j \leq r$. It is obvious that $H \cong GL(T_1) \wr S_r$ (the Wreath product of $GL(T_1)$ and S_r), which consists of all $2r \times 2r$ matrices of the form $\text{diag}(A_1, \dots, A_r)(\alpha \otimes I_2)$, where A_i 's are 2×2 invertible matrices, α is a $r \times r$ permutation matrix and I_2 the 2×2 identity matrix. It is easy to see that H induces an order preserving permutation group acting on $\mathcal{F}'$.

For $0 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq 2$, let $\langle i_1, i_2, \dots, i_r \rangle$ denote the set of all subspaces $A \in L(V)$ such that $\dim(A) = \sum_{j=1}^r i_j$ and the both multi-sets $\{\dim(A \cap T_1), \dots, \dim(A \cap T_r)\}$ and $\{i_1, i_2, \dots, i_r\}$ are identical. Then $\langle i_1, i_2, \dots, i_r \rangle$ is an orbit of H in $\mathcal{F}'$ and the quotient poset is $\widehat{\mathcal{F}}' = \mathcal{F}'/H = \{\langle i_1, i_2, \dots, i_{r-1}, 2 \rangle : 0 \leq i_j \leq 2, \text{ for } 1 \leq j \leq r-1\}$. The ordering of $\widehat{\mathcal{F}}'$ is defined as follows: $\langle i_1, i_2, \dots, i_r \rangle \leq \langle i'_1, i'_2, \dots, i'_r \rangle$ if $i_j \leq i'_j$ for $1 \leq j \leq r$. Under this order $\widehat{\mathcal{F}}'$ is a graded poset. It is obvious that $\widehat{\mathcal{F}}'$ is isomorphic to the

poset $L(r-1, 2)$ which is the set of all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{r-1})$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{r-1} \leq 2$, where λ_i is called the i th part of λ .

Given a $\mu = \langle \mu_1, \mu_2, \dots, \mu_r \rangle \in \widehat{\mathcal{F}}'$, by $w(\mu)$ we denote the cardinality of the orbit μ . Then Theorem 2.1 says that $\widehat{\mathcal{F}}'$ has the NM property if and only if the weighted poset $(\widehat{\mathcal{F}}', w)$ employs the NM property. To express the weight $w(\mu)$ we introduce a few of notations.

For any $\lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_r \rangle \in \widehat{\mathcal{F}}'$, denote $m(\lambda) = (m_0, m_1, m_2)$, where $m_j = m_j(\lambda) = |\{i : \lambda_i = j\}|$. By the definition we immediately have the relation: $r = m_0 + m_1 + m_2$. For convenience, we use $\widehat{\mathcal{F}}'_s$ to denote the rank $\{\langle i_1, i_2, \dots, i_r \rangle : i_1 + i_2 + \dots + i_r = s\}$.

Define $\{1\} = 1$, $\{i\} = 1 + q + \dots + q^{i-1} = \frac{q^i - 1}{q - 1}$ and $\{i\}! = \{1\}\{2\} \cdots \{i\}$. Then $\begin{bmatrix} n \\ m \end{bmatrix} := \frac{\{n\}!}{\{n-m\}!\{m\}!}$, the q -binomial coefficient, is known for the number of the m -subspaces of $V_n(q)$.

Lemma 2.2 *Let $\lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_r \rangle \in \widehat{\mathcal{F}}'$ and $m(\lambda) = (m_0, m_1, m_2)$. Then $w(\lambda) = \begin{pmatrix} r \\ m_0, m_1, m_2 \end{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}^{m_1}$.*

Proof. For any X in the orbit of λ , there are m_0 T'_i 's such that $T_i \cap X = 0$, m_1 T'_i 's such that $\dim(T_i \cap X) = 1$ and m_2 T'_i 's such that $T_i \subset X$. It is obvious that there are $\begin{pmatrix} r \\ m_0, m_1, m_2 \end{pmatrix}$ ways to select T_i satisfying the above condition.

Moreover for each T_i there are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ways to such that $\dim(T_i \cap X) = 1$. Thus

$$w(\lambda) = \begin{pmatrix} r \\ m_0, m_1, m_2 \end{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}^{m_1}.$$

Given $\lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_r \rangle \in \widehat{\mathcal{F}}'_k$ and $m(\lambda) = (m_0, m_1, m_2)$, we define two order-raising operations φ and ψ which map some elements of $\widehat{\mathcal{F}}'_k$ to $\widehat{\mathcal{F}}'_{k+1}$ as follows: $\varphi(\lambda) = \langle \lambda_1, \lambda_2, \dots, \lambda_{m_0} + 1, \dots, \lambda_r \rangle$ if $m_0 \geq 1$ and $\psi(\lambda) = \langle \lambda_1, \lambda_2, \dots, \lambda_{(m_0+m_1)+1}, \dots, \lambda_r \rangle$ if $m_1 \geq 1$. Then $m(\psi(\lambda)) = (m_0, m_1 - 1, m_2 + 1)$ and $m(\varphi(\lambda)) = (m_0 - 1, m_1 + 1, m_2)$. It is easy to see that

$$\nabla(\lambda) = \begin{cases} \{\varphi(\lambda), \psi(\lambda)\} & \text{if } m_0 \geq 1 \text{ and } m_1 \geq 1, \\ \{\varphi(\lambda)\} & \text{if } m_0 \geq 1 \text{ and } m_1 = 0, \\ \{\psi(\lambda)\} & \text{if } m_0 = 0 \text{ and } m_1 \geq 1. \end{cases} \quad (3)$$

Now, we introduce another partial order $\preceq$ on $\widehat{\mathcal{F}}'_s$ for $1 \leq s \leq 2r - 1$. Let $m(\lambda) = (m_0, m_1, m_2)$ and $m(\mu) = (m'_0, m'_1, m'_2)$. We say $\lambda \preceq \mu$ iff $m_0 + \dots + m_i \leq m'_0 + \dots + m'_i$ holds for $i = 0, 1, 2$. Under this partial order $\widehat{\mathcal{F}}'_s$ is a chain.

Lemma 2.3 *Let $\mu, \nu \in \widehat{\mathcal{F}}'_k$ and $\mu \preceq \nu$. Suppose $m(\mu) = (m_0, m_1, m_2)$ and $m(\nu) = (m'_0, m'_1, m'_2)$. Then we have*

(i) If $m_0 \geq 1$, $\frac{w\{\varphi(\mu)\}}{w(\mu)} < \frac{w\{\varphi(\nu)\}}{w(\nu)}$;

(ii) If $m'_1 \geq 1$, $\frac{w\{\psi(\mu)\}}{w(\mu)} > \frac{w\{\psi(\nu)\}}{w(\nu)}$.

Proof. If $m_0 \geq 1$, set $m'_0 = m_0 + 1$, $m'_1 = m_1 - 2$ and $m'_2 = m_1 + 1$. Then we obtain

$$\frac{w(\varphi(\mu))}{w(\mu)} / \frac{w(\varphi(\nu))}{w(\nu)} = \frac{m_0}{m_1 + 1} \cdot \frac{m_1 - 1}{(m_0 + 1)} < 1.$$

Thus the statement (i) follows by induction. If $m'_1 \geq 1$, we can prove $\frac{w\{\psi(\mu)\}}{w(\mu)} > \frac{w\{\psi(\nu)\}}{w(\nu)}$ in a similar way. $\square$

The following trivial result is used repeatedly in the proof.

Lemma 2.4 Let $A_i, B_j > 0$ for $1 \leq i, j \leq k$ and $A_i/B_i \geq A_1/B_1 \geq \ell$ for $2 \leq i \leq k$. Then

$$\frac{A_2 + \cdots + A_k}{B_2 + \cdots + B_k} \geq \frac{A_1 + A_2 + A_3 + \cdots + A_k}{B_1 + B_2 + B_3 + \cdots + B_k} \geq \ell.$$

Suppose $\widehat{\mathcal{F}}'_s = \{\mu_1, \mu_2, \dots, \mu_k\}$ with $\mu_1 \preceq \cdots \preceq \mu_k$. For $1 \leq i \leq k$, define $S_i = \{\mu_i, \dots, \mu_k\}$ and $R_i = \{\mu_1, \dots, \mu_i\}$. Thus we immediately have that $w(S_i)$'s decrease monotonic with i and $w(R_i)$'s increase monotonic with i .

Lemma 2.5 For any rank $\widehat{\mathcal{F}}'_s$ of $\widehat{\mathcal{F}}'$ (where $s < 2r$). Let $\widehat{\mathcal{F}}'_s = \{\mu_1, \mu_2, \dots, \mu_k\}$ and suppose $\mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_k$ where $r = |\widehat{\mathcal{F}}'_s|$. Then we have:

$$(i) \quad \frac{w(\nabla(S_k))}{w(S_k)} \geq \frac{w(\nabla(S_{k-1}))}{w(S_{k-1})} \geq \cdots \geq \frac{w(\nabla(S_1))}{w(S_1)} = \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)};$$

$$(ii) \quad \frac{w(\nabla(R_1))}{w(R_1)} \geq \frac{w(\nabla(R_2))}{w(R_2)} \geq \cdots \geq \frac{w(\nabla(R_k))}{w(R_k)} = \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)}.$$

Proof. Since $s = 2, 2r - 1, 2r$, $\widehat{\mathcal{F}}'_s$ has only one element. Then we only need to consider $3 \leq s \leq 2r - 3$. Given $m(\mu_1) = (m_0, m_1, m_2)$. It is easy to see that

$$\nabla(S_i) = \begin{cases} \nabla(S_{i+1}) \cup \varphi(\mu_i), & i \geq 2 \text{ or } i = 1 \text{ and } m_0 \geq 1; \\ \nabla(S_{i+1}), & i = 1 \text{ and } m_0 = 0. \end{cases}$$

We thus obtain by Lemma 2.3 and Lemma 2.4 that $\frac{w(\nabla(S_i))}{w(\nabla(S_i))} = \frac{w(\psi(\mu_i)) + w(\nabla(S_{i+1}))}{w(\mu_i) + w(S_{i+1})} \leq \frac{w(\nabla(S_{i+1}))}{w(\nabla(S_{i+1}))}$ holds for $i \geq 2$ or $i = 1$ while $m_0 \geq 1$. If $i = 1$ and $m_0 = 0$, then $S_1 = \{\mu_1\} \cup S_2$ and $\nabla(S_1) = \nabla(S_2)$, so $\frac{w(\nabla(S_1))}{w(S_1)} = \frac{w(\nabla(S_2))}{w(\mu_1) + w(S_2)} < \frac{w(\nabla(S_2))}{w(S_2)}$. Thus the statement (i) follows. Similarly, for $i \leq k$, we have

$$\nabla(R_i) = \begin{cases} \nabla(R_{i-1}) \cup \psi(\mu_i), & i < k - 1 \text{ or } i = k \text{ and } s \text{ is odd;} \\ \nabla(R_{i-1}), & i = k \text{ and } s \text{ is even.} \end{cases}$$

Then statement (ii) follows by Lemma 2.3 and Lemma 2.4 $\square$

Lemma 2.6 *Let $\mu_i, \mu_{i+1}, \dots, \mu_j$ be the elements of $\widehat{\mathcal{F}}'_s$ where $1 \leq i \leq j \leq k$. Then $\frac{w(\nabla(\mu_i, \dots, \mu_j))}{w(\{\mu_i, \dots, \mu_j\})} \geq \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)}$.*

Proof. By definition we have that $M = S_i \cap R_j$ and $\nabla(M) = \nabla(S_i) \cap \nabla(R_j)$. It is easy to see that $S_i \cup R_j = \widehat{\mathcal{F}}'_s$ and $\nabla(S_i) \cup \nabla(R_j) = \widehat{\mathcal{F}}'_{s+1}$, which implies that $w(M) = w(S_i) + w(R_j) - w(\widehat{\mathcal{F}}'_s)$ and $w(\nabla(M)) = w(\nabla(S_i)) + w(\nabla(R_j)) - w(\widehat{\mathcal{F}}'_{s+1})$. We thus obtain that

$$\frac{w(\nabla(M))}{w(M)} = \frac{w(\nabla(S_i)) + w(\nabla(R_j)) - w(\widehat{\mathcal{F}}'_{s+1})}{w(S_i) + w(R_j) - w(\widehat{\mathcal{F}}'_s)} \geq \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)}.$$

The last inequality follows from Lemma 2.4 and Lemma 2.5. $\square$

In order to complete the proof of that $\widehat{\mathcal{F}}'$ has the normalized matching property it suffices to prove that for any $1 \leq s \leq 2r - 1$, $\frac{w(\nabla(M))}{w(M)} \geq \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)}$ holds for every $M \subseteq \widehat{\mathcal{F}}'_s$.

Given a subset M of $\widehat{\mathcal{F}}'_s$, set $M = M_1 \cup \dots \cup M_\ell$ ($\ell \geq 1$), where $M_t = \{\mu_{i_t}, \mu_{i_t+1}, \dots, \mu_{j_t}\}$ ($t = 1, 2, \dots, \ell$), $1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \dots < i_\ell \leq j_\ell \leq k$. Then $M_i \cap M_j = \emptyset$ and $\nabla(M_i) \cap \nabla(M_j) = \emptyset$ for $i \neq j$. We thus obtain that $w(M) = w(M_1) + \dots + w(M_\ell)$ and $w(\nabla(M)) = w(\nabla(M_1)) + \dots + w(\nabla(M_\ell))$. Then the inequality $\frac{w(\nabla(M))}{w(M)} \geq \frac{w(\widehat{\mathcal{F}}'_{s+1})}{w(\widehat{\mathcal{F}}'_s)}$ follows from Lemma 2.4 and Lemma 2.6. $\square$

3 Log concavity of $\mathcal{F}'$

We need some definitions before proving the result. For each $X \in \mathcal{F}'$, let $\partial X = (a_1, a_2, \dots, a_r)$ where $a_i = \dim(X \cap T_i)$. For any $X, Y \in \mathcal{F}'$, suppose $\partial X = (a_1, a_2, \dots, a_r)$ and $\partial Y = (b_1, b_2, \dots, b_r)$, we define

$$X \odot Y = (C_1, C_2, \dots, C_r),$$

where C_i be the multiset $\{X \cap T_i, Y \cap T_i\}$ if $a_i + b_i \neq 2$, otherwise C_i is the subspace spanned by $X \cap T_i$ and $Y \cap T_i$. Let $S = \{X \odot Y : X, Y \in \mathcal{F}'\}$, and for each $A = X \odot Y \in S$, let $A_{(i, m-i)} = \{(X, Y) : X \in \mathcal{F}'_i, Y \in \mathcal{F}'_{m-i} \text{ and } X \odot Y = A\}$ where $m = \dim(X) + \dim(Y)$. Clearly, we have $|A_{(i, m-i)}| = |A_{(m-i, i)}|$.

To prove that $\mathcal{F}'$ is log concave, we only need to prove that $\mathcal{F}'_{i-1} \times \mathcal{F}'_{i+1} \leq \mathcal{F}'_i^2$. It is not difficult to see that

$$\mathcal{F}'_i \times \mathcal{F}'_j = \sum_{\substack{A = X \odot Y \in S \\ \dim(X) + \dim(Y) = i+j}} |A_{(i, j)}|.$$

Thus the unimodality of $\{|A_{(i, m-i)}|\}$ on i implies the log concavity of $\mathcal{F}'$. So it remains to verify the unimodality of $|A_{(i, m-i)}|$.

Let B_A be the multiset $\{a_1 + b_1, \dots, a_r + b_r\}$, we call that A is of type 0 if $4 \in B_A$ and $1 \notin B_A$; of type 1 if B_A only contains 2 or 3 besides 0; of type 2 if B_A only contains 2 and 3 besides 0; of type 3 if $1 \in B_A$. It is obviously that for any $A \in S$, B_A must be one type of the above. Suppose $a_i + b_i = 2$ in B_A for some i , if $C_i \neq T_i$, then $X \odot Y = A$ implies that $C_i = X \cap T_i = Y \cap T_i$. Thus we only need to consider the case $C_i = T_i$ when $a_i + b_i = 2$ in B_A .

Let $T_{(k,i)} = \{(X, Y) : X \odot Y = (T_1, \dots, T_k), X = \bigoplus_{i=1}^k (X \cap T_i), \dim(X) = i \text{ and } Y = \bigoplus_{i=1}^k (Y \cap T_i)\}$, we have the following result.

Lemma 3.1 *For each integer $k > 0$, we have $|\{T_{(k,i)}\}|$ is symmetric and unimodal.*

Proof. It is obvious that $|\{T_{(k,i)}\}|$ is symmetric. By induction on k , we are to prove that $|\{T_{(k,i)}\}|$ is unimodal.

If $k = 1$, then $|T_{(1,1)}| = q(q+1) > |T_{(1,0)}| = |T_{(1,2)}| = 1$. Assume that the result is true for k and we consider the case of $k+1$. It is easy to see that

$$|T_{(k+1,i)}| = |T_{(k,i)}| + q(q+1)|T_{(k,i-1)}| + |T_{(k,i-2)}|.$$

Then $|T_{(k+1,i+1)}| - |T_{(k+1,i)}|$

$$\begin{aligned} &= (|T_{(k,i+1)}| + q(q+1)|T_{(k,i)}| + |T_{(k,i-1)}|) \\ &\quad - (|T_{(k,i)}| + q(q+1)|T_{(k,i-1)}| + |T_{(k,i-2)}|) \\ &= (|T_{(k,i+1)}| - |T_{(k,i)}|) + q(q+1)(|T_{(k,i)}| - |T_{(k,i-1)}|) + (|T_{(k,i-1)}| - |T_{(k,i-2)}|). \end{aligned}$$

By the hypothesis of induction and the symmetry of $\{T_{(k,i)}\}$ we have that $|T_{(k+1,i+1)}| > |T_{(k+1,i)}|$ if $0 \leq i \leq k$. Thus the result is true for $k+1$ and we complete the proof. $\square$

Lemma 3.2 *Let $A = X \odot Y \in S$ and $m = \dim(X) + \dim(Y)$. If A is of type 0, then $|A_{(2,m-2)}| \leq |A_{(3,m-3)}| \leq \dots \leq |A_{(\ell,m-\ell)}|$, where $\ell = \lfloor m/2 \rfloor$.*

Proof. Suppose B_A contains s 3's, and we use $A_{(i,m-i)}^s$ to replace $A_{(i,m-i)}$. It is easy to see that

$$|A_{(i,m+3-i)}^{s+1}| = |A_{(i-1,m-i+1)}^s| + |A_{(i-2,m-i+2)}^s|. \quad (4)$$

Now we use induction on s to prove the result. By Lemma 3.1 we know that the result is true for $s = 0$. Assume that the result is true for s and consider the case of $s+1$. By (4) we have

$$|A_{(i+1,m+2-i)}^{s+1}| - |A_{(i,m+3-i)}^{s+1}| = |A_{(i,m-i)}^s| - |A_{(i-2,m-i+2)}^s|.$$

If $i \leq \lfloor (m+1)/2 \rfloor$, by the hypothesis of induction and the symmetry of $|A_{(i,m-i)}|$ on i we have $|A_{(i+1,m+2-i)}^{s+1}| - |A_{(i,m+3-i)}^{s+1}| \geq 0$. Thus the result is true for $s+1$ and we derive the result. $\square$

Lemma 3.3 *Let $A = X \odot Y \in S$ and $m = \dim(X) + \dim(Y)$. If A is of type 1, then $|A_{(2, m-2)}| \leq |A_{(3, m-3)}| \leq \cdots \leq |A_{(k, m-k)}|$, where $k = \lfloor m/2 \rfloor$.*

Proof. Now we distinguish two cases to consider B_A .

Case 1. $B_A = \{3\}$. Suppose B_A contains k 3's. It is easy to see that $|A_{(k+i, m-k-i)}| = \binom{k}{i}$. By the unimodality of binomial coefficients the statement follows.

Case 2. $B_A = \{2\}$. Suppose B_A contains k 2's. It is easy to see that $A_{(i, k)}$ has the following recursion:

$$|A_{(i, 2(k+1)-i)}| = \begin{cases} |T_{(k, i-2)}| + q(q+1)|A_{(i-1, 2k-i+1)}| + |A_{(i, 2k-i)}| & \text{if } i < k+1, \\ 2|T_{(k, k-1)}| + q(q+1)|A_{(k, k)}| & \text{if } i = k+1, \end{cases}$$

where $|A_{(i, 2k-i)}| = 0$ if $i < 2$.

Now, we use induction on k to prove that $|A_{(2, 2k-2)}| \leq |A_{(3, 2k-3)}| \leq \cdots \leq |A_{(k, k)}|$.

If $k = 2$, the statement is easy to verify. Assume that the result is true for k and consider the case of $k+1$.

If $i < k+1$, we have $|A_{(i, 2k+2-i)}| - |A_{(i-1, 2k+3-i)}|$

$$\begin{aligned} &= (|T_{(k, i-2)}| + q(q+1)|A_{(i-1, k)}| + |A_{(i, 2k-i)}|) \\ &\quad - (|T_{(k, i-2)}| + q(q+1)|A_{(i-2, 2k+2-i)}| + |A_{(i-1, 2k-i+1)}|) \\ &= (|T_{(k, i-2)}| - |T_{(k, i-3)}|) + q(q+1)(|A_{(i-1, k+1-i)}| - |A_{(i-2, 2k+2-i)}|) \\ &\quad + (|A_{(i, 2k-i)}| - |A_{(i-1, 2k-i+1)}|) \geq 0. \end{aligned}$$

Since each of the three terms in the sum are nonnegative by induction and the unimodality of binomial coefficients.

If $i = k+1$, $|A_{(k+1, k+1)}| - |A_{(k, k+2)}|$

$$\begin{aligned} &= (2|T_{(k, k-1)}| + q(q+1)|A_{(k, k)}|) \\ &\quad - (|T_{(k, k-2)}| + q(q+1)|A_{(k-1, k+1)}| + |A_{(k, k)}|) \\ &= (|T_{(k, k-1)}| - |T_{(k, k-2)}|) + q(q+1)(|A_{(k, k)}| - |A_{(k-1, k+1)}|) \\ &\quad + (|T_{(k, k-1)}| - |A_{(k-1, k+1)}|). \end{aligned}$$

It is obvious that $|T_{(k, k-1)}| \geq |A_{(k-1, k+1)}|$ when B_A only contains 2 besides 0. Then by induction we have $|A_{(k+1, k+1)}| - |A_{(k, k+2)}| \geq 0$, as desired. $\square$

For each $A \in S$ and be type 2, suppose B_A contains s 2's and k 3's, let $A'_{(i, k, s)} = \{(X, Y) : X \in \mathcal{F}_i, Y \in \mathcal{F}_{m-i} \text{ and } X \odot Y = A\}$ such that if $a_i + b_i = 2$ in B_A , then either $T_i \subseteq X$ or $T_i \cap X = \emptyset$.

Lemma 3.4 *Let $A \in S$ and be of type 2, then $|A'_{(2, k, s)}| \leq |A'_{(3, k, s)}| \leq \cdots \leq |A'_{(r, k, s)}|$ where $r = \lfloor 2s + 3k/2 \rfloor$.*

Proof. It is easy to see that $|A'_{(k+1,k,s)}| = k$, $|A'_{(k,k,s)}| = 1$ and $|A'_{(i,k,s)}| = 0$ for $i < k$. At first, we give a recursion of $|A'_{(i,k+1,s)}|$:

$$|A'_{(i,k+1,s)}| = \begin{cases} k+1 & \text{if } i = k+2; \\ |A'_{(i-2,k,s)}| + |A'_{(i-1,k,s)}| & \text{if } k+3 \leq i. \end{cases}$$

In fact, suppose $a_j + b_j = 3$, for $i \geq k+3$ and each $(X, Y) \in A'_{(i, 3k+3+2s-i)}$. Then we have $(X/T_j, Y/T_j) \in A'_{(i-1, 3k+2s+1-i)}$ if $\dim(T_j \cap X) = 1$, and $(X/T_j, Y/T_j) \in A'_{(i-2, 3k+2s+2-i)}$ otherwise. Thus $|A'_{(i,k+1,s)}| = |A'_{(i-2,k,s)}| + |A'_{(i-1,k,s)}|$.

Now we use induction on k to prove that

$$|A'_{(2,k,s)}| \leq |A'_{(3,k,s)}| \leq \cdots \leq |A'_{(r,k,s)}|.$$

If $k = 0$, by Lemma 3.3 the result is true. Assume that the result is true for k and we consider the case of $k+1$. If $k+3 \leq i \leq \lfloor (3k+2s+3)/2 \rfloor$, $|A'_{(i,k+1,s)}| - |A'_{(i-1,k+1,s)}|$

$$\begin{aligned} &= (|A'_{(i-2,k,s)}| + |A'_{(i-1,k,s)}|) - (|A'_{(i-3,k,s)}| + |A'_{(i-2,k+1,s)}|) \\ &= |A'_{(i-1,k,s)}| - |A'_{(i-3,k,s)}|. \end{aligned}$$

Then by the induction hypothesis and the symmetry of $A_{(i,k,s)}$ on i we have $|A'_{(i,k+1,s)}| - |A'_{(i-1,k+1,s)}| \geq 0$ and the statement follows. $\square$

Lemma 3.5 Let $A = X \odot Y \in S$ and $m = \dim(X) + \dim(Y)$. If A is of type 2, then $|A_{(2,m-2)}| \leq |A_{(3,m-3)}| \leq \cdots \leq |A_{(k,m-k)}|$, where $k = \lfloor m/2 \rfloor$.

Proof. Suppose B_A contains k 3's and s 2's. For $0 \leq t \leq s$, define

$$A_{(i,k,t)} = \{(X, Y) : X \in \mathcal{F}_i, Y \in \mathcal{F}_{m-i} \text{ and } X \odot Y = A\}$$

such that there are t T'_i 's either $T_i \subseteq X$ and $T_i \cap Y = \{0\}$ or $T_i \subseteq Y$ and $T_i \cap X = \{0\}$. By lemma 3.4, we have $|A_{(2,k,t)}| \leq |A_{(3,k,t)}| \leq \cdots \leq |A_{(\lfloor m/2 \rfloor, k, t)}|$. Since $|A_{(i,m-i)}| = \sum_{t=0}^s |A_{(i,k,t)}|$, then we have $|A_{(2,m-2)}| \leq |A_{(3,m-3)}| \leq \cdots \leq |A_{(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor)}|$. Thus we complete the proof. $\square$

Lemma 3.6 Let $A = X \odot Y \in S$ and $m = \dim(X) + \dim(Y)$. If A is of type 3, then $|A_{(2,m-2)}| \leq |A_{(3,m-3)}| \leq \cdots \leq |A_{(k,m-k)}|$, where $k = \lfloor m/2 \rfloor$.

Proof. Suppose B_A contains k 1's, we use $A_{(k,i,m)}$ to denote $A_{(i,m)}$. Now we use induction on k to prove the result. It is easy to see that $|A_{(k+1,i,m+1)}| = |A_{(k,i,m)}| + |A_{(k,i-1,m)}|$. If $k = 0$, the result holds true by Lemmas 3.2, 3.3 and 3.5. Assume that the result holds for k and consider the case of $k+1$. For $2 \leq i \leq \lfloor m-1/2 \rfloor$,

$$\begin{aligned} &|A_{(k+1,i+1,m+1)}| - |A_{(k+1,i,m+1)}| \\ &= (|A_{(k,i+1,m)}| + |A_{(k,i,m)}|) - (|A_{(k,i,m)}| + |A_{(k,i-1,m)}|) \\ &= |A_{(k,i+1,m)}| - |A_{(k,i-1,m)}|. \end{aligned} \tag{5}$$

By the induction hypothesis and the symmetry of $|A_{(k,i,m)}|$ on i we have $(6) \geq 0$ and the statement follows. $\square$

By Lemma 3.2-3.6, we know that for any $A = X \odot Y \in S$ and $\dim(X) + \dim(Y) = 2i$, $|A_{(i-1,i+1)}| \leq |A_{(i,i)}|$. Thus we obtain

$$\mathcal{F}'_{i-1} \times \mathcal{F}'_{i+1} = \sum_{\substack{A=X \odot Y \in S \\ \dim(X)+\dim(Y)=2i}} |A_{(i-1,i+1)}| \leq \sum_{\substack{A=X \odot Y \in S \\ \dim(X)+\dim(Y)=2i}} |A_{(i,i)}| = \mathcal{F}'_i{}^2.$$

Then the statement follows.

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Received: May 23, 2006