Abstract. Let $R$ be a Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ the field of rational numbers) and $\delta$ be a derivation of $R$. An ideal $P$ is an associated prime ideal of differential operator ring $R[x, \delta]$ if and only if $P = (P \cap R)[x, \delta]$ and $P \cap R$ is an associated prime ideal of $R$.

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1. Introduction

A ring $R$ means with identity and any $R$-module unitary. Let $R$ be a ring. The set of prime ideals of $R$ is denoted by $Spec(R)$, the set of associated prime ideals of $R$ is denoted by $Ass(R)$ and the set of minimal prime ideals of $R$ is denoted by $Min.Spec(R)$. $Assas(M)$ denotes the assassinator of a uniform $R$-module $M$ and for any subset $J$ of an $R$-module $M$, the annihilator of $J$ is denoted by $Ann(J)$. $C(0)$ denotes the set of regular elements of a ring $R$ and $C(I)$ denotes the set of elements regular modulo $I$, where $I$ is an ideal of $R$. For any two ideals $I, J$ of a ring $R$; $I \subset J$ means that $I$ is strictly contained in $J$. $\mathbb{R}$ and $\mathbb{Q}$ denote the fields of real numbers and rational numbers respectively. $\mathbb{Z}$ denotes the ring of integers unless otherwise stated.

We know that a Lie algebra over a field $K$ is a vector space $L$ over $K$ equipped with a non-associative product $[..]$ satisfying the usual distributive laws and the rules $[xx] = 0$, $[x[yz]] + [y[zx]] + [z[xy]] = 0$. For example $\mathbb{R}^3$ equipped with
the usual cross product is a real Lie algebra. One can see that an associative
K-algebra equipped with the operation \([.,.]\) such that \([x,y] = xy - yx\) becomes
a Lie algebra over K.

Conversely starting with a Lie algebra, one can construct an associative K-
algebra \(U(L)\) using the elements of L as generators, together with the relations
\(xy-yx = [xy]\) for all \(x, y \in L\). The algebra \(U(L)\) is called the Universal
enveloping algebra of L.

The simplest Lie algebra \(L\) with a non-zero product is two-dimensional, with
a basis \(\{x, y\}\) such that \([yx] = x\). The elements of \(U(L)\) in this case can be
put in the form \(\sum f_i(x)y^i\), \(i = 1, 2, ..., n\); where each \(f_i(x)\) is a polynomial
in the variable \(x\). In \(U(L)\) the relation \([yx] = x\) becomes \([y,x] = x\) and so it
follows easily that \([y, f(x)] = x(d/dx)(f(x))\) for all polynomials \(f(x)\). We note
that \(U(L)\) is very similar to the first complex Weyl algebra \(A_1(C)\). In \(A_1(C)\)
we have \([D,f(x)] = (d/dx)(f(x))\).

Keeping all this in mind, we start with a ring \(R\) and a map \(\delta : R \rightarrow R\)
which is a derivation and then construct a large ring \(T\) using an indeterminate
\(y\) such that \([y,a] = \delta(a)\) for all \(a \in R\). The elements of \(T\) look like differential
operators \(\sum \delta a_i\) on \(R\), except that it may be possible for \(\sum \delta a_i\) to be zero
operator without all the coefficients \(a_i\) being zero. Thus the elements \(\sum y^i a_i\)
in \(T\) are called formal differential operators and \(T\) is called formal differential
operator ring.

By Hilbert Basis Theorem, we see that formal differential operator rings
with Noetherian coefficient rings are Noetherian and we shall view them as
representative analogs of enveloping algebras.

Let now \(R\) be a Noetherian Q-algebra and \(\delta\) be a derivation of \(R\). In this
paper we deal with the differential operator ring \(R[x, \delta]\). We denote this ring by
\(D(R)\). We mention that the coefficients of polynomials are taken to be on right,
and thus \(D(R) = \{\sum x^i a_i, a_i \in R\}\) subject to the relation \(ax = xa + \delta(a)\).
Some authors take coefficients on the left.

We give a structure of associated and minimal prime ideals of the differential
operator ring \(D(R)\), where \(R\) is right Noetherian Q-algebra. This is proved
in 2.9. Before giving the structure of associated prime ideals of \(D(R)\), we first
prove for a Noetherian Q-algebra \(R\), a result of Seidenberg, namely Theorem
(1) of [5] and a result of Gabriel, namely Lemma (3.4) of [2] in one go. This is
proved in 2.7.

2. Associated Prime Ideals

We begin this section with the following definition:

**Definition 2.1.** Let \(R\) be ring and \(\delta\) be a derivation of \(R\). Then \(D(R) = R[x, \delta]\) is the usual differential operator ring. Let \(I\) be a \(\delta\)-invariant ideal of \(R\). Then \(D(I) = I[x, \delta]\).

**Proposition 2.2.** Let \(R\) be a Noetherian Q-algebra and \(\delta\) be a derivation of
\(R\). Then \(e^{t\delta}\) is an automorphism of \(T = R[[t]]\).
**Proof.** The proof is same as given by Seidenberg in [5] and a sketch in the non-commutative case is provided by Blair and Small in [1].

**Remark 2.3.** Let $R$ be a Noetherian ring and $\delta$ be a derivation of $R$. Let $I$ be an ideal of $R$. Then we know that $I.R[[t]] = \{b_0 + tb_1 + t^2b_2 + ..., \text{ with } b_i \in I\}$ and denote it by $I[[t]]$.

**Lemma 2.4.** Let $R$ be a Noetherian $Q$-algebra and $\delta$ be a derivation of $R$. Then an ideal $I$ of $R$ is $\delta$-invariant if and only if $I.R[[t]]$ is $e^{t\delta}$-invariant.

**Proof.** Let $T = R[[t]]$. Let $IT$ be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in IT$, which implies that $e^{t\delta}(a) \in IT$; i.e. $a + t\delta(a) + (t^2\delta^2/2!)(a) + ... \in IT$. Therefore we have $\delta(a) \in I$.

Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum t^i a_i \in IT$. Then $e^{t\delta}(f) = f + t\delta(f) + (t^2\delta^2/2!)(f) + ... \in IT$ as $\delta(a_i) \in I$. Thus we have $e^{t\delta}(IT) \subseteq IT$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(IT) = IT$.

We now quote the following Proposition, the proof of which is routine.

**Proposition 2.5.** Let $R$ be a ring and $T = R[[t]]$. Then:

1. $Q \in \text{Ass}(R)$ implies that $QT \in \text{Ass}(T)$.
2. $P \in \text{Ass}(T)$ implies that $P \cap R \in \text{Ass}(R)$ and $P = (P \cap R)T$.

**Proposition 2.6.** Let $R$ be a ring and $T = R[[t]]$. Then:

1. $P \in \text{Min.Spec}(T)$ implies that $P \cap R \in \text{Min.Spec}(R)$ and $P = (P \cap R)T$.
2. $Q \in \text{Min.Spec}(R)$ implies that $QT \in \text{Min.Spec}(T)$.

**Proof.** (1) Let $P \in \text{Min.Spec}(T)$. Then $P \cap R \in \text{Spec}(R)$. Suppose $(P \cap R) \notin \text{Min.Spec}(R)$, and $S \subseteq P \cap R$ is a minimal prime ideal of $R$. Then $ST \subseteq (P \cap R)T \subseteq P$, which is a contradiction as $ST \in \text{Spec}(R)$. Therefore $(P \cap R) \in \text{Min.Spec}(R)$. Now it is easy to see that $(P \cap R)T = P$.

(2) Let $Q \in \text{Min.Spec}(R)$. Then $QT \in \text{Spec}(T)$. Suppose $QT \notin \text{Min.Spec}(T)$, and $J \subseteq QT$ is a minimal Prime ideal of $T$. Then $(J \cap R) \subseteq QT \cap R = Q$ which is a contradiction as $(J \cap R) \in \text{Spec}(R)$. Therefore $QT \in \text{Min.Spec}(T)$.\[\square\]

**Theorem 2.7.** Let $R$ be a Noetherian $Q$-algebra and $\delta$ be a derivation of $R$. Let $P \in (\text{Ass}(R) \cup \text{Min.Spec}(R))$. Then $\delta(P) \subseteq P$.

**Proof.** Let $T = R[[t]]$. Now by 2.2 $e^{t\delta}$ is an automorphism of $T$. Let $P \in (\text{Ass}(R) \cup \text{Min.Spec}(R))$. Then by 2.5 and 2.6 $PT \in (\text{Ass}(T) \cup \text{Min.Spec}(T))$. Therefore there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{n\delta}(PT) = PT$. Now $R$ is a $Q$-algebra implies that $e^{n\delta}(PT) = PT$ and, therefore 2.4 implies that $\delta(P) \subseteq P$.\[\square\]

**Proposition 2.8.** Let $R$ be a semiprime Noetherian ring. Let $\delta$ be a derivation of $R$. If $f \in D(R)$ is a regular element. Then there exists $g \in D(R)$ such that $gf$ has leading co-efficient regular in $R$. (The proof of this Proposition is obvious and is left to the reader).
Theorem 2.9. Let $R$ be a Noetherian $Q$-algebra and $\delta$ be a derivation of $R$. Then:

1. $P \in \text{Ass}(D(R))$ if and only if $P = D(P \cap R)$ and $(P \cap R) \in \text{Ass}(R)$.
2. $P \in \text{Min.Spec}(D(R))$ if and only if $P = D(P \cap R)$ and $(P \cap R) \in \text{Min.Spec}(R)$.

Proof. (1) Let $P \in \text{Ass}(R)$. Then $\delta(P) \subseteq P$ by 2.7. Let $P = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$. Since $cP = 0$, so $\delta^{k}(c)P = 0$ for all $k \geq 0$. Now let $h = \sum x^{i}b_{i} \in D(R)$ with leading coefficient $b_{0}$. Then for all $r \in R$, $cRhP = 0$; i.e. $cRhP = 0$ and therefore $cRh.D(R) = 0$. Now by (14.2.5) (ii) of [4] $D(P) \subseteq \text{Spec}(D(R))$. Suppose $D(P) \neq \text{Ann}(ch.D(R))$. There exists an ideal $K$ of $D(R)$ such that $D(P) \subset K$ and $K = \text{Ann}(ch.D(R))$, $ch \neq 0$. Therefore by 2.8 there exists $g = \sum x^{i}a_{i} \in K$ with leading coefficient $a_{t}$ such that $a_{t} \in C(P)$. Now $ch.D(R)K = 0$, which implies that $chRg = 0$. So for all $r \in R$, $chrg = 0$; i.e. $c(x^{i}b_{i} + ... + b_{0})r(x^{i}a_{t} + ... + a_{0}) = 0$. Therefore we have $(x^{a+t}c + ... + a_{t}(c)b_{u}ra_{t} + ... + cb_{0}r_{0} = 0$. Thus $cb_{u}r_{0} = 0$ for all $r \in R$; i.e. $cb_{u}r_{u} = 0$. Therefore we have $a_{t} \in \text{Ann}(cb_{u}R)$, which is a contradiction as $a_{t} \in C(P)$. Therefore $D(P) = \text{Ann}(ch.D(R))$ for all $h \in D(R)$. Hence $D(P) = \text{Assas}(ch.D(R))$.

Conversely suppose that $P \in \text{Ass}(D(R))$. Choose $f = \sum x^{i}a_{i}$ (with leading coefficient $a_{t}$) of least degree such that $P = \text{Ann}(f.D(R)) = \text{Assas}(f.D(R))$. Now $a_{t}R(P \cap R) = 0$ for all $i$, $1 \leq i \leq n$. Let $P_{1} \in \text{Ass}(a_{n}R)$. Then there exists $s \in R$ such that $a_{n}s \neq 0$ and $P_{1} = \text{Ann}(a_{n}sR) = \text{Assas}(a_{n}sR)$. Now $\delta(P_{1}) \subseteq P_{1}$ by 2.7 and by above paragraph $D(P_{1}) \subseteq \text{Ass}(D(R))$. Now $(a_{n}sRP_{1}) = 0$ so $d^{t}(a_{n}s)RP_{1} = 0$ for all integers $t \geq 1$ and for any $h = \sum x^{i}b_{i} \in D(R)$, $a_{n}sRhP_{1} = 0$, which implies that $a_{n}sR.D(R).P_{1} = 0$. Therefore we have $P_{1} \subseteq (P \cap R)$. Also $(P \cap R) \subseteq P_{1}$, as $a_{n}sR(P \cap R) = 0$. Thus we have $P_{1} = P \cap R$, which means that $D(P_{1}) = D(P \cap R) \in \text{Ass}(D(R))$. Therefore as in first paragraph $D(P \cap R) = P$.

(2). Let $P_{1} \in \text{Min.Spec}(R)$. Then 2.7 implies that $\delta(P_{1}) \subseteq P_{1}$. So by (14.2.5) (ii) of [4] $D(P_{1}) \subseteq \text{Spec}(D(R))$. Suppose $D(P_{1}) \notin \text{Min.Spec}(D(R))$, and $P_{2} \subseteq D(P_{1})$ is a minimal prime ideal of $D(R)$. Then $P_{2} = D(P_{2} \cap R) \subseteq D(P_{1}) \subseteq \text{Min.Spec}(D(R))$, which implies that $(P_{2} \cap R) \subset P_{1}$, which is a contradiction as $(P_{2} \cap R) \in \text{Spec}(R)$. Therefore $D(P_{1}) \in \text{Min.Spec}(D(R))$.

Conversely if $P \in \text{Min.spec}(D(R))$, then $(P \cap R) \in \text{Spec}(R)$ by Lemma (2.21) of [3] and $D(P \cap R) \in \text{Spec}(D(R))$. Therefore $D(P \cap R) = P$. We now show that $(P \cap R) \in \text{Min.spec}(R)$. Suppose $(P \cap R) \notin \text{Min.spec}(R)$, and $P_{1} \subset (P \cap R)$ is a minimal prime ideal of $R$. Then $D(P_{1}) \subset D(P \cap R)$ and as in first paragraph $D(P_{1}) \in \text{Spec}(D(R))$, which is a contradiction. Hence $(P \cap R) \in \text{Min.spec}(R)$.

References


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