On the Order of Schur Multipliers of Finite Abelian $p$-Groups

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Abstract

Let $G$ be a finite $p$-group of order $p^n$ with $|M(G)| = p^{n(n-1)/2-t}$, where $M(G)$ is the Schur multiplier of $G$. Ya.G. Berkovich, X. Zhou, and G. Ellis have determined the structure of $G$ when $t = 0, 1, 2, 3$. In this paper, we are going to find some structures for an abelian $p$-group $G$ with conditions on the exponents of $G, M(G)$, and $S_2M(G)$, where $S_2M(G)$ is the metabelian multiplier of $G$.

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1 Introduction and Preliminaries

Let $G$ be any group with a presentation $G \cong F/R$, where $F$ is a free group. Then the Baer invariant of $G$ with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]}$$

where $V$ is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of $F$ and
\[ [RV^*F] = \langle v(f_1, ..., f_{i-1}, f_i r, f_{i+1}, ..., f_n) v(f_1, ..., f_i, ..., f_n)^{-1} \mid r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in N \rangle. \]

In particular, if \( V \) is the variety of abelian groups, \( A \), then the Baer invariant of the group \( G \) will be \((R \cap F^*)/[R, F]\) which is isomorphic to the well-known notion the Schur multiplier of \( G \), denoted by \( M(G) \) (see [5,6] for further details).

If \( V \) is the variety of polynilpotent groups of class row \((c_1, ..., c_t)\), \( N_{c_1, c_2, ..., c_t} \), then the Baer invariant of a group \( G \) with respect to this variety is as follows:

\[
N_{c_1, c_2, ..., c_t} M(G) = \frac{R \cap \gamma_{c_t+1} \circ ... \circ \gamma_{c_1+1}(F)}{[R, c_1 F, c_2 \gamma_{c_1+1}(F), ..., c_t \gamma_{c_{t-1}+1} \circ ... \circ \gamma_{c_1+1}(F)]},
\]

where \( \gamma_{c_t+1} \circ ... \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\cdots(\gamma_{c_1+1}(F))\cdots)) \) are the term of iterated lower central series of \( F \). See [4] for the equality

\[
[R N_{c_1, c_2, ..., c_t}^* F] = [R, c_1 F, c_2 \gamma_{c_1+1}(F), ..., c_t \gamma_{c_{t-1}+1} \circ ... \circ \gamma_{c_1+1}(F)].
\]

In particular, if \( c_i = 1 \) for \( 1 \leq i \leq t \), then \( N_{c_1, c_2, ..., c_t} \) is the variety of solvable groups of length at most \( t \geq 1, S_t \).

In 1956, J.A. Green [3] showed that the order of the Schur multiplier of a finite \( p \)-group of order \( p^n \) is bounded by \( p^{\frac{n(n-1)}{2}} \), and hence equals to \( p^{\frac{n(n-1)}{2}} \cdot t \), for some nonnegative integer \( t \). In 1991, Ya.G. Berkovich [1] has determined all finite \( p \)-groups \( G \) for which \( t = 0, 1 \). The groups for which \( t = 0 \) are exactly elementary abelian \( p \)-groups, and the groups for which \( t = 1 \) are cyclic groups of order \( p^2 \) or the nonabelian group of order \( p^3 \) with exponent \( p > 2 \). In 1994, X. Zhou [7] found all finite \( p \)-groups for \( t = 2 \). He showed that these groups are the direct product of two cyclic groups of order \( p^2 \) and \( p \) or the direct product of a cyclic group of order \( p \) and the nonabelian group of order \( p^3 \) and exponent \( p > 2 \) or the dihedral group of order 8. G. Ellis [2] determined all finite \( p \)-groups \( G \) with \( t = 0, 1, 2, 3 \) in a quite different method to that of [1] and [7] as follows:

**Theorem 1.1** ([2]). Let \( G \) be a group of prime-power order \( p^n \). Suppose that \( M(G) \) has order \( p^{\frac{n(n-1)}{2}} \cdot t \). Then \( t \geq 0 \) and

(i) \( t = 0 \) if and only if \( G \) is elementary abelian;

(ii) \( t = 1 \) if and only if \( G \cong \mathbb{Z}_{p^2} \) or \( G \cong E_1 \);

(iii) \( t = 2 \) if and only if \( G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}, G \cong D \) or \( G \cong \mathbb{Z}_p \times E_1 \);

(iv) \( t = 3 \) if and only if \( G \cong \mathbb{Z}_{p^3}, G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p, G \cong D \times \mathbb{Z}_{p^2}, G \cong E_2, G \cong Q \) or \( G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times E_1 \).

Here \( \mathbb{Z}_{p^m} \) denotes the cyclic group of order \( p^m \), \( D \) denotes the dihedral group of order 8, \( Q \) denotes the quaternion group of order 8, \( E_1 \) denotes the extra special group of order \( p^3 \) with odd exponent \( p \), and \( E_2 \) denotes the extra special
group of order \( p^3 \) with odd exponent \( p^2 \).

Now, in this paper, we are going to find some structures for the \( p \)-group \( G \) when \( G \) is abelian and \( |\Phi(G)| = p^a \) with conditions on the exponents of \( G, M(G), \) and \( S_2 M(G) \). The following useful theorem of I. Schur is frequently used in our method.

**Theorem 1.2** (I. Schur [5]). Let \( G \cong Z_{n_1} \oplus Z_{n_2} \oplus ... \oplus Z_{n_k} \), where \( n_{i+1} | n_i \) for all \( i \in 1, 2, ..., k-1 \) and \( k \geq 2 \), and let \( Z_{n_i}^{(m)} \) denote the direct product of \( m \) copies of \( Z_{n_i} \). Then

\[
M(G) \cong Z_{n_2} \oplus Z_{n_3}^{(2)} \oplus ... \oplus Z_{n_k}^{(k-1)}
\]

**Remark 1.3.** Let \( G \) be an abelian group with a free presentation \( F/R \). Since \( F' \leq R \), \( N_{c_1} M(G) = \gamma_{c_1+1}(F)/[R, c_1 F] \). Now, we can consider \( \gamma_{c_1+1}(F)/[R, c_1 F] \) as a free presentation for \( N_{c_1} M(G) \) and hence

\[
N_{c_2} M(N_{c_1} M(G)) = \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[R, c_1 F, c_2 \gamma_{c_1+1} F]}.
\]

Therefore by (1) we have

\[
N_{c_1,c_2} M(G) = N_{c_2} M(N_{c_1} M(G)).
\]

By continuing the above process we can show that

\[
N_{c_1,c_2,...,c_t} M(G) = N_{c_t} M(\cdots N_{c_2} M(N_{c_1} M(G))\cdots).
\]

In particular, if \( c_1 = c_2 = 1 \), then we have \( S_2 M(G) = M(M(G)) \).

## 2 Main Results

Through out the paper we assume that \( G \) is an abelian \( p \)-group of order \( p^n \) with \( |M(G)| = p^{\frac{n(a+1)}{2} - t} \).

**Lemma 2.1.** Let \( \Phi(G) \), the Frattini subgroup of \( G \), be of order \( p^a \). Then \( n = (a(a+1) + 2t + 2m)/2a \), for some \( m \in N_0 \).

**Proof.** Let \( G = Z_{p^{\alpha_1}} \oplus Z_{p^{\alpha_2}} \oplus ... \oplus Z_{p^{\alpha_{n-a}}} \), where \( \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-a} \).

By Theorem 1.2 , \( M(G) \cong Z_{p^{\alpha_2}} \oplus Z_{p^{\alpha_3}}^{(2)} \oplus ... \oplus Z_{p^{\alpha_{n-a}}}^{(n-a-1)} \) and so \( |M(G)| = \)
\( p^{\alpha_2 + 2\alpha_3 + \ldots + (n-a-1)\alpha_{n-a}} \). But \( M(G) \) has order \( p^{\frac{n(a-1)}{2} - t} \). Therefore

\[
\frac{n(n-1)}{2} - t = \alpha_2 + 2\alpha_3 + \ldots + (n-a-1)\alpha_{n-a} \\
\geq 1 + 2 + \ldots + (n-a-1) \\
= \frac{(n-a)(n-a-1)}{2} \\
= \frac{n^2 - (2a+1)n + a(a+1)}{2}.
\]

Hence \( 2an \geq 2t + a(a+1) \), and the result holds.

**Lemma 2.2.** With the assumption and notation of the previous lemma we have the following inequalities for the exponent of \( G \),

\[
p^{a-m+1} \leq \exp(G) \leq p^{a+1}.
\]

**Proof.** Clearly \( G/\Phi(G) \) is an elementary abelian \( p \)-group of order \( p^{n-a} \) and so \( \exp(G) \leq p^{a+1} \).

For the other inequality let \( G \cong \mathbb{Z}_{p^\alpha_1} \oplus \mathbb{Z}_{p^\alpha_2} \oplus \ldots \oplus \mathbb{Z}_{p^\alpha_{n-a}} \), where \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-a} \geq 1 \). Then similar to the proof of previous lemma we have

\[
\frac{n(n-1)}{2} - t = \alpha_2 + 2\alpha_3 + \ldots + (n-a-1)\alpha_{n-a} \\
\geq \alpha_2 + (2 + 3 + \ldots + n - a - 1) \\
\geq \frac{2\alpha_2 - 2 + n^2 - (2a+1)n + a(a+1)}{2}.
\]

Therefore \( n \geq \frac{a(a+1) + 2\alpha_2 - 2 + 2t}{2a} \) and hence by Lemma 2.1 we have \( \alpha_2 \leq m + 1 \). Now, suppose by contrary \( \exp(G) = p^{a-k+1} \), where \( k > m \), then \( \alpha_3 \geq 2 \). Thus by Theorem 1.2 we have

\[
\frac{n(n-1)}{2} - t = \alpha_2 + 2\alpha_3 + \ldots + (n-a-1)\alpha_{n-a} \\
= (\alpha_2 + \alpha_3 + \ldots + \alpha_{n-a}) + \alpha_3 + 2\alpha_4 + \ldots + (n-a-2)\alpha_{n-a} \\
\geq (n - a + k - 1) + (2 + 2 + 3 + \ldots + n - a - 2) \\
= n - a + k + \frac{(n-a-2)(n-a-1)}{2}.
\]

Hence \( n \geq \frac{2k + a(a+1) + 2 + 2t}{2a} > \frac{2m + a(a+1) + 2 + 2t}{2a} \) which is a contradiction by Lemma 2.1.
Theorem 2.3. With the above notation and assumptions, let $G$ be of exponent $p^{a-m+1}$. Then $G \cong \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^{m+1}} \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-a} \geq 1$. By the proof of previous lemma we have $\alpha_2 \leq m + 1$. If $\alpha_2 \leq m$, then $\alpha_3 \geq 2$ and we have

$$\frac{n(n-1)}{2} - t = \alpha_2 + 2\alpha_3 + \ldots + (n-a-1)\alpha_{n-a}$$

$$= (\alpha_2 + \alpha_3 + \ldots + \alpha_{n-a}) + \alpha_3 + 2\alpha_4 + \ldots + (n-a-2)\alpha_{n-a}$$

$$\geq (n-a + m - 1) + (2 + 3 + \ldots + n-a-2)$$

$$\geq n-a + m + \frac{(n-a-2)(n-a-1)}{2}.$$ 

Therefore $n \geq \frac{2m+a(a+1)+2+2t}{2a}$ which is a contradiction by Lemma 2.1. Hence the result holds.

Theorem 2.4. Further to the previous notation and assumptions, let $m = k + s$ $(k, s \in \mathbb{N}_0)$, $exp(G) = p^{a-k+1}$, and $exp(M(G)) + exp(S_2M(G)) = p^{k+r}$. Then

$$G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{k+r-s}} \oplus \mathbb{Z}_{p^{h_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{h_f}} \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p,$$

where $x = k - s + 2r - 3 + 3(h_1 - 1) + \ldots + (f + 2)(h_f - 1)$, $h_1 \geq h_2 \geq \ldots \geq h_f \geq 2$, and $f \leq -r + 2$.

Proof. Let $G \cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^a} \oplus \ldots \oplus \mathbb{Z}_{p^{a-n-a}}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-a}$. By Theorem 1.2 and Remark 1.3 it is easy to see that $exp(M(G)) = p^{a_2}$, $exp(S_2M(G)) = p^{a_3}$, and so by hypothesis $\alpha_2 + \alpha_3 = k + r$, and $\alpha_1 = a - k + 1$. If $\alpha_3 = 1$, then it is easy to see that $s = 0$ and so $exp(G) = a - m + 1$. Hence by Theorem 2.3 $G \cong \mathbb{Z}_{p^{a-m+1}} \oplus \mathbb{Z}_{p^{m+1}} \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$.

Now, we can assume that $\alpha_3 \geq 2$ and hence $G$ may have the following structure:

$$G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{k+r-s}} \oplus \mathbb{Z}_{p^{h_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{h_f}} \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p,$$

where $f \geq 0$ and $h_1 \geq h_2 \geq \ldots \geq h_f \geq 2$. If $f \geq 1$ since $G$ is a group of order $p^n$, we have
\[(a-k+1)+(k+r)+(h_1+...+h_f)+(n-a-f-3) = n \] so \[h_1+...+h_f = -r+f+2\] and hence \[h_f = -r+f+2-h_1-...-h_{f-1}.\] But \[h_1 \geq h_2 \geq ... \geq h_f \geq 2,\] so that \[-r+f+2 \geq 2f\] and so \[0 \leq f \leq -r+2.\] Now, by Theorem 1.2 we have

\[
\frac{n^2 - n - 2t}{2} = z + (1 + 2 + ... + n - a - 1),
\]

where

\[
z = x + 2k + 2r - 2x + 3h_1 + 4h_2 + ... + (f + 2)h_f - (1 + 2 + ... + f + 2).
\]

Thus

\[
n = \frac{2z + a(a + 1) + 2t}{2a}.
\]

On the other hand by the hypothesis \[n = \frac{2(k+s)+a(a+1)+2t}{2a},\] hence we have \[z = k + s,\] and the result follows.

**Corollary 2.5.** With the notation and assumptions of previous theorem we have

(i) \[G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-3}} \oplus \mathbb{Z}_{p^{3+1}} \oplus \mathbb{Z}_p \oplus ... \oplus \mathbb{Z}_p, \quad \text{if } r = 2;\]

(ii) \[G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-1}} \oplus \mathbb{Z}_{p^{2}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p}, \quad \text{if } r = 1;\]

(iii) if \(r = 0,\) then

\[
G \cong \begin{cases} 
\mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-4}} \oplus \mathbb{Z}_{p^{4}} \oplus \mathbb{Z}_{p^{2}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p} & n-a-3\text{-copies} \\
\mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-3}} \oplus \mathbb{Z}_{p^{3}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p} & n-a-4\text{-copies} 
\end{cases};
\]

(iv) if \(r = -1,\) then

\[
G \cong \begin{cases} 
\mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-8}} \oplus \mathbb{Z}_{p^{8}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p} & n-a-4\text{-copies} \\
\mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-5}} \oplus \mathbb{Z}_{p^{5}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p} & n-a-5\text{-copies} \\
\mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^{k-7}} \oplus \mathbb{Z}_{p^{7}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p} & n-a-6\text{-copies} 
\end{cases}.
\]

**Proof.** i) If \(r = 2,\) then \(f = 0.\) Therefore

\[
G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p} \oplus \mathbb{Z}_{p^{k+r-x}} \oplus \mathbb{Z}_{p} \oplus ... \oplus \mathbb{Z}_{p},
\]
where $x = k - s + 1$. Hence the result follows.

ii) If $r = 1$, then $f = 0, 1$. If $f = 0$, then $n = a - k + 1 + k + 1 + n - a - 3$ which is a contradiction. Then $f = 1$ and

$$G \cong \mathbb{Z}_{p^{a-k+1}} \oplus \mathbb{Z}_{p^x} \oplus \mathbb{Z}_{p^{k+r-x}} \oplus \mathbb{Z}_{p^h_1} \oplus \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p,$$

where $h_1 = -r + f + 2 = 2$ and $x = k - s + 2 + 6 - (1 + 2 + 3) = k - s + 2$. Hence the result follows.

iii) If $r = 0$, then $f = 0, 1, 2$. If $f = 0$, then $h_1 + h_2 + \ldots + h_f = 0$ but we have $h_1 + h_2 + \ldots + h_f = -r + f + 2 = 2$ which is a contradiction. If $f = 1$, then $x = k - s + 4$ and if $f = 2$, then $x = k - s + 3$.

iv) By a routine calculation similar to (ii) the result holds.

Note that we can continue the above corollary for other integers $r < -1$, but with a boring calculations.

**References**


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