

# Complex Inversion Formula for Stieltjes and Widder Transforms with Applications

A. Aghili and A. Ansari

Department of Mathematics, Faculty of Sciences  
University of Guilan, P.O. Box 1841, Rasht, Iran  
armanaghili@yahoo.com, alireza\_1038@yahoo.com

## Abstract

In this article, we derive a complex inversion formula and some new theorems related to Widder-transform defined in [2],[3],we give also some identities involving  $L_2$ -transform and the new transforms. Some constructive examples are also given.

**Mathematics Subject Classification:** Primary 44A10, Secondary 44A35

**Keywords:** Widder potential transform,  $L_2$ -transform, Laplace transform, Stieltjes transform, Widder transforms

## 1 Introduction

The basic aim of the transform method is to convert a given problem into one that is easier to solve. Integral transform methods provide effective ways to solve a variety of problems arising in the engineering and physical sciences. The Laplace - transform is by far the most prominent in application. Many other Transforms have been developed, but most have limited applicability. This work also contains some discussion on several other integral transforms that have been used recently successfully in the solution of certain problems and in other applications. Included in this category are,  $L_2$ -transform, Stieltjes-transform, Widder potential integral transform ( [2],[3] ). The Laplace-transform of the function is defined as

$$L\{f(t)\} = F(s) = \int_0^{+\infty} \exp(-st)f(t)dt \quad (1-1)$$

The inverse laplace-transform is given by the integral relationship

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds \quad (1-2)$$

where  $\sigma$  is any suitably chosen large positive constant. This improper integral is a contour integral taken along the vertical line  $s = \sigma + i\tau$  in complex  $s$ -plane. for more detail see[4]. The Laplace-type integral transform called  $L_2$ -transform where the  $L_2$ -transform is defined as

$$L_2\{f(t); s\} = \int_0^{\infty} t \exp(-s^2 t^2) f(t) dt \quad (1-3)$$

If we make a change of variable in the right-hand side of the above integral (1-3) we get,

$$L_2\{f(t); s\} = \frac{1}{2} \int_0^{\infty} e^{-ts^2} f(\sqrt{t}) dt \quad (1-4)$$

we have the following relationship between the Laplace-transform and the  $L_2$ -transform

$$L_2\{f(t); s\} = \frac{1}{2} L\{f(\sqrt{t}); s^2\} \quad (1-5)$$

First, we calculate  $L_2$  -transform of some special functions.

**Example 1.1 :**

$$L_2\left\{\frac{\operatorname{erf}(kt)}{t}\right\} = \int_0^{+\infty} t e^{-s^2 t^2} \frac{\operatorname{erf}(kt)}{t} dt = \frac{1}{s\sqrt{\pi}} \arctan \frac{k}{s}$$

*Solution :* Setting  $s = 1$  in the above integral, we obtain,

$$\int_0^{+\infty} e^{-t^2} \operatorname{erf}(kt) dt = \frac{1}{\sqrt{\pi}} \arctan(k)$$

**Example 1.2:** Show that

$$\int_0^{+\infty} t^{2m} e^{-mt^2} dt = \frac{\Gamma(m + \frac{1}{2})}{2a^{m+\frac{1}{2}}} = \frac{1 \times 3 \times \dots \times (2m - 1)\sqrt{\pi}}{2^m a^{m+\frac{1}{2}}} \quad (1-6)$$

*Solution:* By definition ,  $L_2\{t^n; s\} = \int_0^{\infty} t^{n+1} \exp(-s^2 t^2) dt$   
the integral on the right-hand side may be evaluated by changing the variable of the integration from  $t$  to  $u$  where,  $s^2 t^2 = u$

$$L_2\{t^n; s\} = \int_0^{\infty} \left(\frac{u^{\frac{1}{2}}}{s}\right)^{n+1} e^{-u} \frac{du}{2su^{\frac{1}{2}}} = \frac{1}{2s^{n+2}} \int_0^{\infty} u^{\frac{n}{2}} e^{-u} du \quad (1-7)$$

using some Gamma function's relation in (1-8), we obtain

$$L_2\{t^n; s\} = \frac{\Gamma(\frac{n}{2} + 1)}{2s^{n+2}}$$

In the above relationship, we set  $n = 2k - 1$ ,  $s^2 = m$  and simplifying to yield,

$$\int_0^{+\infty} t^{2m} e^{-mt^2} dt = \frac{\Gamma(m + \frac{1}{2})}{2a^{m+\frac{1}{2}}} = \frac{1 \times 3 \times \dots \times (2m - 1)\sqrt{\pi}}{2^m a^{m+\frac{1}{2}}}$$

**Example 1.3:** We have the following

1.

$$L_2\{e^{-at}; s\} = \frac{1}{2s^2} - \frac{a\sqrt{\pi}}{4s^3} \exp\left(\frac{a^2}{4s^2}\right) \operatorname{Erfc}\left(\frac{a}{2s}\right) \tag{1-8}$$

2.

$$L_2\{\operatorname{Erf}(at); s\} = \frac{a}{2s^2\sqrt{s^2 + a^2}} \tag{1-9}$$

*Solution 1:* see [1].

*Solution 2:* By definition,

$$\begin{aligned} L_2\{\operatorname{Erf}(at); s\} &= \int_0^\infty t \exp(-s^2t^2) \operatorname{Erf}(at) dt \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty t \exp(-s^2t^2) \left( \int_0^{at} \exp(-u^2) du \right) dt \end{aligned} \tag{1-10}$$

changing the order of in integration in (1-11), we get

$$\begin{aligned} L_2\{\operatorname{Erf}(at); s\} &= \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-u^2) \left( \int_{\frac{u}{a}}^\infty t \exp(-s^2t^2) dt \right) du \\ &= \frac{1}{s^2\sqrt{\pi}} \int_0^\infty \exp\left(-u^2\left(1 + \frac{s^2}{a^2}\right)\right) du \end{aligned} \tag{1-11}$$

if we set  $u\left(1 + \frac{s^2}{a^2}\right)^{1/2} = v$ , one has

$$L_2\{\operatorname{Erf}(at); s\} = \frac{a}{2s^2\sqrt{s^2 + a^2}}$$

*Note :* In the above relationship, if we set  $s = a = 1$ , we get the following

$$\int_0^\infty te^{-t^2} \operatorname{erf}(t) dt = \frac{1}{2\sqrt{2}}$$

**Lemma 1.1 :** Let us assume that  $L_2\{f(t); s\} = F(s)$  then,

$$L_2\left\{f\left(\frac{t}{a}\right); s\right\} = a^2 F(as)$$

*Proof :* By definition,  $L_2\left\{f\left(\frac{t}{a}\right); s\right\} = \int_0^\infty t \exp(-s^2t^2) f\left(\frac{t}{a}\right) dt$   
 setting  $\frac{t}{a} = u \Rightarrow t = au$ , we get

$$\begin{aligned} L_2\left\{f\left(\frac{t}{a}\right); s\right\} &= a^2 \int_0^\infty u \exp(-s^2a^2u^2) f(u) du \\ &= a^2 \int_0^\infty u \exp(-(as)^2u^2) f(u) du = a^2 F(as) \end{aligned} \tag{Q.E.D}$$

**Lemma 1.2 :** Let us assume that  $L_2\{f(t); s\} = F(s)$ , then

$$L_2\left\{\int_0^t uf(u)du, s\right\} = -\frac{1}{2s^2}F(s)$$

*Proof:* By definition of  $L_2$ -transform, we have

$$L_2\left\{\int_0^t uf(u) du, s\right\} = \int_0^\infty e^{-s^2t^2} t \left(\int_0^t uf(u) du\right) dt$$

changing the order of integration, to get

$$L_2\left\{\int_0^t uf(u)du\right\} = \int_0^{+\infty} uf(u)du \int_u^{+\infty} te^{-s^2t^2} dt = \frac{-1}{2s^2} \int_0^\infty ue^{-s^2u^2} f(u)du = -\frac{1}{2s^2}F(s)$$

**Example 1.4 :** It is easy to show that  $L_2\left\{\frac{\cos at}{t}\right\} = \frac{\sqrt{\pi}}{2s}e^{-\frac{a^2}{4s^2}}$  therefore, by the above theorem we obtain

$$L_2\left\{\int_0^t u\left(\frac{\cos au}{u}\right)du\right\} = L_2\left\{\int_0^t (\cos au)du\right\} = L_2\left\{\frac{1}{a} \sin au\right\} = \frac{\sqrt{\pi}}{4s^3}e^{-\frac{a^2}{4s^2}}$$

on the other hand , one gets

$$L_2\left\{\int_0^t (\cos au)du\right\} = \frac{1}{2s^2}\left\{\frac{\sqrt{\pi}}{2s}e^{-\frac{a^2}{4s^2}}\right\} = \frac{\sqrt{\pi}}{4s^3}e^{-\frac{a^2}{4s^2}}$$

which is the same result.

**Example 1.5 :**

$$\begin{aligned} L_2\left\{\int_0^t uJ_0(au) du, s\right\} &= \int_0^\infty e^{-s^2t^2} t \left(\int_0^t uJ_0(au) du\right) dt \\ &= \frac{1}{2s^2}\left\{\frac{1}{2s^2}e^{-\frac{a^2}{4s^2}}\right\} \end{aligned}$$

or,

$$L_2\left\{\int_0^t uJ_0(au)du\right\} = \frac{1}{4s^4}e^{-\frac{a^2}{4s^2}}$$

Now, if we sets  $s = 1, a = 2$ , one has the following relationship

$$\int_0^\infty te^{-t^2} dt \int_0^t uJ_0(au)du = \frac{1}{4e}$$

## 2 Complex Inversion Formula for $L_2$ -transform

**Main Theorem :** Let  $F(\sqrt{s})$  is analytic function of  $s$  (assuming that  $s=0$  is not a branch point) except at finite number of poles each of which lies to the left of the vertical line  $\text{Re } s = c$  and if  $F(\sqrt{s}) \rightarrow 0$  as  $s \rightarrow \infty$  through the left plane  $\text{Re } s \leq c$ , suppose that :

$$L_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2)f(t)dt = F(s)$$

then,

$$\begin{aligned} L_2^{-1}\{F(s)\} = f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2} dt \\ &= \sum_{k=1}^m [Res \{2F(\sqrt{s})e^{st^2}\}, s = s_k] \end{aligned}$$

*Proof :* We may use the same procedure as complex inversion formula for Laplace Transform [1].

**Example 2.1 :** By using complex inversion formula for  $L_2$ -Transform, show that

$$L_2^{-1}\left[\frac{1}{2} \arctan \frac{a}{s^2}\right] = \frac{\sin at^2}{t^2}$$

*Solution:* Letting,  $F(s) = \frac{1}{2}Arctg\frac{a}{s^2}$  then we have ,

$$F(\sqrt{s}) = \frac{1}{2} \arctan \frac{a}{s}$$

therefore,  $s = 0$  is a singular point(essential singularity not branch point). After using the above mentioned complex inversion formula, we obtain the original function as following,

$$f(t) = Res\left\{\frac{1}{s} \arctan \frac{a}{s} \exp st^2; s = 0\right\} = b_{-1}$$

where  $b_{-1}$  is the coefficient of the term  $\frac{1}{s}$  in the Laurent expansion of  $2F(\sqrt{s}) \exp st^2$ . Therefore we get the following relationship,

$$2F(\sqrt{s}) \exp st^2 = \frac{1}{s} \left[1 + (st^2) + \frac{(st^2)^2}{2!} + \dots\right] \left[a - \frac{a^3}{3s^3} + \frac{a^5}{5s^5} - \frac{a^7}{7s^7!} + \dots\right]$$

from the above expansion we obtain,

$$\begin{aligned} f(t) = b_{-1} &= \left[a - \frac{a^3t^4}{3!} + \frac{a^5t^8}{5!} - \frac{a^7t^{10}}{7!} + \dots\right] \\ &= \frac{1}{t^2} \left[at^2 - \frac{a^3t^6}{3!} + \frac{a^5t^8}{5!} - \frac{a^7t^{10}}{7!} + \dots\right] = \frac{\sin at^2}{t^2} \end{aligned}$$

**Example 2.2 :** Using Complex inversion formula, show that

$$L_2^{-1}\left[\frac{\Gamma(n+1)}{s^{2(n+1)}}\right] = t^{2n}$$

*Solution* : We have,  $F(\sqrt{s}) = \frac{\Gamma(n+1)}{s^{n+1}}$  on using complex inversion formula, we get

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2} dt = \frac{1}{n!} \lim_{s \rightarrow 0} \left\{ \frac{d^n}{ds^n} [\Gamma(n+1)e^{st^2}] \right\} \\ &= t^{2n} \end{aligned}$$

**Example 2.3** : Show that

$$L_2^{-1} \left[ \frac{\Gamma(n)}{2(s^2 + a^2)^n} \right] = t^{2(n-1)} e^{-a^2 t^2}$$

*Solution* : Letting  $F(s) = \frac{\Gamma(n)}{2(s^2+a^2)^n}$  which implies,  $F(\sqrt{s}) = \frac{\Gamma(n)}{2(s+a^2)^n}$  upon using Complex inversion formula for  $L_2$ -Transform, we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2 \left\{ \frac{\Gamma(n)}{2(s+a^2)^n} \right\} e^{st^2} ds \Rightarrow f(t) = \frac{\Gamma(n)}{(n-1)!} \lim_{s \rightarrow -a^2} \left\{ \frac{d^{n-1}}{ds^{n-1}} e^{st^2} \right\} \\ f(t) &= \lim_{s \rightarrow -a^2} (t^2)^{n-1} e^{st^2} \Rightarrow f(t) = t^{2(n-1)} e^{-a^2 t^2} \end{aligned}$$

*Note* : In the above example,  $s = -a^2$  is a pole of order  $n$ .

**Lemma 2.1**: Show that,

$$\int_0^\infty t \exp(-t^2) (\ln t)^2 dt = 2 + \frac{5}{2}\gamma + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12},$$

*Proof*: By definition, we have  $L_2\{t^n; s\} = \frac{\Gamma(\frac{n+1}{2})}{2s^{\frac{n+1}{2}}}$  or,

$$L_2\{t^\lambda; s\} = \int_0^\infty t \exp(-s^2 t^2) t^\lambda dt = \frac{\Gamma(\frac{\lambda}{2} + 1)}{2s^{\lambda+2}}$$

If we differentiate the above relation w.r.t  $\lambda$  (using Leibnitz's rule), we obtain

$$\int_0^\infty t \exp(-s^2 t^2) t^\lambda \ln t dt = \frac{d}{d\lambda} \left[ \frac{\Gamma(\frac{\lambda}{2} + 1)}{2s^{\lambda+2}} \right]$$

or,

$$\int_0^\infty t \exp(-s^2 t^2) t^\lambda \ln t dt = \frac{1}{2s^2} \left[ \frac{\Gamma(\frac{\lambda+2}{2})}{s^{2\lambda}} \right] \left[ \frac{\Gamma'(\frac{\lambda+2}{2})}{\Gamma(\frac{\lambda+2}{2})} - 2 \ln s \right] \quad (2-1)$$

if we set  $\lambda = 0$  and assuming  $\Gamma'(1) = -\gamma$ , we get

$$\int_0^\infty t \exp(-s^2 t^2) \ln t dt = l_2[\ln t] = -\frac{\gamma + \ln s^2}{2s^2}$$

If we differentiate relation (2-1) w.r.t  $\lambda$  for second time and setting  $\lambda = 0, s = 1$ , one has

$$\int_0^\infty t \exp(-t^2)(\ln t)^2 dt = \frac{4 + 5\gamma + \Gamma''(1)}{2}$$

but, we have  $\Gamma''(1) = \gamma^2 + \frac{\pi^2}{6}$ , therefore, one gets

$$\int_0^\infty t \exp(-t^2)(\ln t)^2 dt = 2 + \frac{5}{2}\gamma + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12}$$

### 3 Stieltjes Transform

Stieltjes Transform is defined as follows

$$\rho\{f(t), y\} = \int_0^\infty \frac{f(t)dt}{t + y}$$

It is well-known that the second iterate of the laplace transform is the Stieltjes-Transform; [2],[5],that is,

$$L^2\{f(t); y\} = L\{L\{f(t); u\}, y\} = \rho\{f(t), y\} = F(y) \tag{3-1}$$

**Example 3.1 :** For  $f(t) = \delta(t - a)$ , we can verify the above relationship we have

$$\rho\{\delta(t - a), y\} = \int_0^\infty \frac{\delta(t - a)dt}{t + y} = \frac{1}{a + y}$$

on the other hand, second iterate of the Laplace transform of  $\delta(t - a)$  is,  $L\{\delta(t - a), u\} = e^{-au}$  and  $L\{e^{-au}, y\} = \frac{1}{y+a}$  Q.E.D

**Example 3.2 :** show that  $\rho\{t^{\mu-1}, y\} = \frac{\pi y^{\mu-1}}{\sin \mu\pi}$  ( $0 < \mu < 1$ )

*Solution :* Upon using relationship (3-1)

$$L^2\{f(t); y\} = L\{L\{f(t); u\}, y\} = \rho\{f(t), y\} = \int_0^{+\infty} \frac{t^{\mu-1}}{t + y} dt$$

$$\begin{aligned} L^2\{f(t); y\} &= L\{L\{f(t); u\}, y\} = L\{L\{t^{\mu-1}; u\}, y\} = L\left\{\frac{\Gamma(\mu)}{u^\mu}, y\right\} \\ &= \Gamma(\mu)L\{u^{-\mu}, y\} = \Gamma(\mu)\gamma(1 - \mu)y^{\mu-1} = \frac{\pi}{\sin \mu\pi}y^{\mu-1} \end{aligned}$$

#### 3.1 Widder potential Transform

is defined as follows

$$P\{f(t), y\} = \int_0^\infty \frac{tf(t)dt}{t^2 + y^2}$$

It is well-known that the second iterate of the  $L_2$ - transform is the Widder potential transform; [3]that is,

$$L_2^2\{f(t); y\} = L_2\{L_2\{f(t); u\}, y\} = P\{f(t), y\} = \int_0^{+\infty} \frac{tf(t)}{t^2 + y^2} dt$$

**Example 3.1 :**

$$P\{\delta(t - a), y\} = \int_0^{\infty} \frac{t\delta(t - a)dt}{t^2 + y^2} = \frac{a}{a^2 + y^2}$$

**Lemma 3.2 :** We have the following relationship,

$$\frac{1}{y}\rho\{\sin t, y\} = P\left\{\frac{e^{-t}}{t}, y\right\}$$

*Solution :* Let us assume that  $\Gamma_R$  be the closed contour composed of a quarter circle  $z = Re^{i\theta}$ ,  $0 \leq \theta \leq \frac{\pi}{2}$  in first quadrant and two line segments  $[0, R], [0, iR]$  traversed in counter clockwise, and let  $F(z) = \frac{e^{iz}}{z+a}$ ,  $a > 0$ , then by Cauchy theorem, one has

$$\oint_{\Gamma_R} \frac{e^{iz}}{z+a} dz = 0$$

Now, we expand the above integral to obtain

$$\int_0^R \frac{e^{ix}}{x+a} dx + \int_0^{\frac{\pi}{2}} \frac{e^{i(iR^{i\theta})}}{Re^{i\theta} + a} d\theta + \int_R^0 \frac{e^{i(iy)}}{iy+a} idy = 0$$

taking limit as R tends to infinity, to get

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx + \lim_{R \rightarrow +\infty} \int_0^{\frac{\pi}{2}} \frac{e^{i(iR^{i\theta})}}{Re^{i\theta} + a} d\theta + \int_{+\infty}^0 \frac{e^{i(iy)}}{iy+a} idy = 0$$

the second integral tends to zero ( in absolute value) therefore, one has the following

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx + \int_{+\infty}^0 \frac{e^{i(iy)}}{iy+a} idy = 0$$

or,

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx = \int_0^{+\infty} \frac{ae^{-y}}{y^2 + a^2} dy$$

taking imaginary part of both sides of the above relationship, to yield

$$\frac{1}{a} \int_0^{+\infty} \frac{\sin x}{x+a} dx = \int_0^{+\infty} \frac{x \frac{e^{-x}}{x}}{x^2 + a^2} dx$$

in terms of Widder potential and Stiltjes Transforms, we have

$$\frac{1}{a} \rho\{\sin x, a\} = P\left\{\frac{e^{-x}}{x}, a\right\}$$

### 3.2 Complex inversion formula for Stieltjes Transform and Widder transform

The inverse Stieltjes Transform is given by the following integral relationship

$$f(t) = \rho^{-1}\{F(y)\} = \frac{1}{(2\pi i)^2} \int_{\lambda-i\infty}^{\lambda+i\infty} \left( \int_{\sigma-i\infty}^{\sigma+i\infty} F(y)e^{yx} dy \right) e^{xt} dx \quad (3-2)$$

where  $\sigma, \lambda$  are any suitably chosen large positive constants. This improper integral is a contour integral taken along the vertical lines  $y = \sigma + i\tau$  and  $x = \lambda + i\eta$  in complex-plane.

**Example 3.2:** Show that

$$\rho^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, t\} = t^{-\frac{3}{4}}$$

*Proof :* Since Stieltjes transform is second iterate of Laplace transform, we can re-write Relation (3-1) as follows

$$f(t) = L^{-1}\{L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\}, t\}$$

using the fact that  $t^\nu = L^{-1}\left\{\frac{\Gamma(\nu+1)}{s^{\nu+1}}\right\}$  hence,  $L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\} = \frac{\pi \sqrt{2}}{\Gamma(\frac{3}{4})} x^{-\frac{1}{4}}$  finally,

$$f(t) = L^{-1}\{L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\}, t\} = L^{-1}\left\{\frac{\pi \sqrt{2}}{\Gamma(\frac{3}{4})} x^{-\frac{1}{4}}, t\right\} = \frac{\pi \sqrt{2}}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} t^{-\frac{3}{4}} = t^{-\frac{3}{4}} \quad (3-3)$$

### 3.3 Complex inversion formula for Widder Transform

**Theorem 3.2 :** Let  $P\{f(t), y\} = \int_0^\infty \frac{tf(t)dt}{t^2+y^2} = F(y)$  then we have,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2F(\sqrt{y})e^{yw^2} dy \right]_{w \rightarrow \sqrt{w}} e^{wt^2} dw$$

Provided that, all integrals involved are convergent.

*Proof:* That is the natural consequence of definition of Widder Potential transform and complex inversion formula for  $L_2$ - transform .

**Lemma 3.3 :** We have  $P\left\{\frac{\ln x}{x}, y\right\} = \frac{\pi \ln y}{y} y > 0$

*Proof:* we take the function  $f(z) = \frac{\log z}{z^2+y^2}$  where  $z = re^{i\theta}, \theta \in ]-\frac{\pi}{2}, \frac{3\pi}{2}]$

We consider the closed path  $\Gamma$  of integration consists of, two segments  $[-R, -\varepsilon], [\varepsilon, R]$  of the x-axis together with the upper semi-circles  $C_\varepsilon : z = \varepsilon e^{i\theta}$

and  $C_R : z = re^{i\theta}$  with  $0 < \theta < \pi$ . We consider the branch of  $\log z$  which is analytic on  $\Gamma$  and its interior, hence, so is  $f(z)$ . By residue theorem one has,

$$\oint_{\Gamma} f(z)dz = 2\pi i \operatorname{Res}\{f(z), yi\} = \frac{\pi \ln y}{y} + \frac{i\pi^2}{2y}$$

or,

$$\int_{-R}^{-\varepsilon} f(z)dz + \int_{-C_{\varepsilon}} f(z)dz + \int_{\varepsilon}^R f(z)dz + \int_{C_R} f(z)dz = \frac{\pi \ln y}{y} + \frac{i\pi^2}{2y}$$

If we take the limit as  $\varepsilon \rightarrow 0, R \rightarrow +\infty$ , integrals along two semi-circles tend to zero, therefore we get

$$P.V\left\{\int_{-\infty}^0 \frac{\ln|x| + i\pi}{x^2 + y^2} dx + \int_0^{+\infty} \frac{\ln x}{x^2 + y^2} dx\right\} = \frac{\pi \ln y}{y} + i\frac{\pi^2}{2y},$$

taking real part of the two sides, yields

$$\int_0^{+\infty} \frac{\ln x}{x^2 + y^2} dx = \frac{\pi \ln y}{2y} = P\left\{\frac{\ln x}{x}, y\right\} \quad Q.E.D$$

## 4 Conclusion

In this article, author provided complex inversion formula for Stieltjes and Widder Potential integral transforms. Some identities involving these new transforms are given. We conclude by remarking that many identities involving various integral transforms can be obtained and some other infinite integrals can be evaluated by applying the results considered here.

## References

- [1] A. Aghili et al: An inversion technique for  $L_2$ -transform with applications, *Int. Contemp. Math. Sciences*, Vol. 2, 2007, No. **28**, 1387-1394
- [2] D. Brown et al: Identities for the exponential integral and the complementary error Transforms, *Applied Mathematics and computations* 182, 2006, 1377-1384.
- [3] D. Brown et al: Identities for the  $E_{2,1}$  - Transforms and their applications, *Applied Mathematics and computations* 182, 2006, 1377-1384.
- [4] A. D. Osborn: Complex analysis with applications, Mc.Graw Hill 1999.

Received: October 26, 2007