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# An Inversion Technique for $E_{2,1}$-Transform with Applications

**A. Aghili and F. Safarian**

Department of Mathematics, Faculty of Sciences
University of Guilan, P.O. Box 1841, Rasht, Iran
armanaghili@yahoo.com, safarian85@ yahoo.com

**Abstract**

In this article, authors derive a complex inversion formula and some new theorems related to Widder Potential and $E_{2,1}$ - transform defined in [2], some identities and constructive examples involving these transforms are also given.

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**Keywords:** $E_{2,1}$-transform, Laplace transform, Heaviside's function

## 1 Introduction

In this work, we study the widder potential and $E_{2,1}$-transforms, which, while closely related to $L_2$-transform where the $L_2$- transform is introduced by Yurekli and Sadek [3]. We state and prove the inversion formula for the above mentioned transforms and develop some of the more important technique associated with its definition as a contour integral. We do not attempt to present the basic theorems in their most general forms. Proofs of theorems are provided, when feasible. The following Laplace-type integral transform called $L_2$-transform where the $L_2$-transform

$$L_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2) f(t)dt. \tag{1-1}$$

If we make a change of variable in the right-hand side of the above integral (1-1), we get,

$$L_2\{f(t); s\} = \frac{1}{2}\int_0^\infty e^{-ts^2} f(\sqrt{t})dt, \tag{1-2}$$

we have the following relationship between the Laplace -transform and the $L_2$-transform

$$L_2\{f(t); s\} = \frac{1}{2}L\{f(\sqrt{t}); s^2\}. \tag{1-3}$$

First, we calculate $L_2$-transform of some special functions.

**Example 1.1:**

$$L_2\{\frac{\cos ax}{x}\} = \int_0^{+\infty} xe^{-s^2x^2}\frac{\cos ax}{x}dx = \int_0^{+\infty} e^{-s^2x^2}\cos ax dx$$
$$= \frac{\sqrt{\pi}}{2s}e^{-\frac{a^2}{4s^2}}. \tag{1-4}$$

Setting $s = 1$ in the above integral, we obtain,

$$\int_0^{+\infty} e^{-x^2}\cos ax\, dx = \frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4}},$$

we differentiate (1-4) w.r.t $a$ , we get the following

$$\int_0^{+\infty} xe^{-s^2x^2}\sin ax\, dx = \frac{a\sqrt{\pi}}{4s^3}e^{-\frac{a^2}{4s^2}}.$$

In terms of $L_2$-transform

$$L_2\{\sin ax, s\} = \frac{a\sqrt{\pi}}{4s^3}e^{-\frac{a^2}{4s^2}} \tag{1-5}$$

**Example1.2:** Show that

$$L_2\{H(t-a); s\} = \frac{1}{2s^2}e^{-s^2a^2}.$$

*Solution:*

$$L_2\{H(t-a); s\} = \int_0^{\infty} t\exp(-s^2t^2)H(t-a)dt = \int_a^{\infty} t\exp(-s^2t^2)dt$$

$$= -\frac{1}{2s^2}\, e^{-s^2t^2}\Big|_a^{\infty} = \frac{1}{2s^2}e^{-s^2a^2} \quad (t > a).$$

**Example1.3:** Show that

$$L_2\{t^{\lambda}; s\} = \frac{\Gamma(\frac{\lambda}{2}+1)}{2s^{\lambda+2}}. \tag{1-6}$$

*Solution:* By definition , $L_2\{t^{\lambda}; s\} = \int_0^{\infty} t^{\lambda+1}\exp(-s^2t^2)dt$ the integral on the right-hand side may be evaluated by changing the variable of the integration from $t$ to $u$ where, $s^2t^2 = u$

$$L_2\{t^{\lambda}; s\} = \int_0^{\infty}\left(\frac{u^{\frac{1}{2}}}{s}\right)^{\lambda+1} e^{-u}\frac{du}{2su^{\frac{1}{2}}} = \frac{1}{2s^{\lambda+2}}\int_0^{\infty} u^{\frac{\lambda}{2}}e^{-u}du, \tag{1-7}$$

using some Gamma function's relation in (1-7), we have

$$L_2\{t^{\lambda}; s\} = \frac{\Gamma(\frac{\lambda}{2}+1)}{2s^{\lambda+2}}.$$

## 2 Complex Inversion Formula for $L_2$ -Transform

**Main Theorem** :Let $F(\sqrt{s})$ is analytic function of s ( assuming that $s = 0$ is not a branch point) except at finite number of poles each of which lies to the left of the vertical line Re s = c and if $F(\sqrt{s}) \to 0\ as\ s \to \infty$ through the left plane $Res \leq c$ , suppose that :

$$L_2\{f(t); s\} = \int_0^\infty t \exp(-s^2 t^2) f(t) dt = F(s),$$

then, [1]

$$L_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s}) e^{st^2} ds = \sum_{k=1}^{m} [\ Res\ \{2F(\sqrt{s}) e^{st^2}\}, s = s_k].$$

*Proof :* By definition, we have $L_2\{f(t); s\} = \int_0^\infty t \exp(-s^2 t^2) f(t) dt = F(s)$ now, let $s^2 = r$ , then we have

$$F(\sqrt{r}) = \int_0^{+\infty} t\ \exp(-rt^2) f(t) dt,$$

now, letting $t^2 = x$ , to yield

$$F(\sqrt{r}) = \frac{1}{2} \int_0^{+\infty} \exp(-rx) f(\sqrt{x}) dx,$$

or,

$$2F(\sqrt{r}) = \int_0^{+\infty} \exp(-rx) f(\sqrt{x}) dx = L\{f(\sqrt{x})\}.$$

By complex inversion formula for Laplace-Transforms one has,

$$f(\sqrt{x}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{r}) \exp(rx) dr.$$

If we set $t^2 = x$ , we obtain

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{r}) \exp(rt^2) dr.$$

If we replace $r$ by $s$ , one has finally

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s}) \exp(st^2) ds. \qquad Q.E.D$$

**Lemma 2.1:** We have the following relationship

$$\frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-(u+\frac{\lambda^2}{4u})} \frac{du}{\sqrt{u}} = e^{-\lambda}.$$

*Proof:* Let us assume that, $g(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-(u+\frac{\lambda^2}{4u})} \frac{du}{\sqrt{u}}$, then, application of Leibnitz's rule leads to, $g(\lambda) = -g'(\lambda)$ with $g(0) = 1$ which gives finally, $g(\lambda) = e^{-\lambda}$ .

**Example 2.1 :**

$$L_2^{-1}\{\frac{e^{-s}}{2s^2}\} = Erfc(\frac{1}{2t}).$$

*Solution:* We have $F(s) = \frac{e^{-s}}{2s^2}$ , which gives $2F(\sqrt{s}) = \frac{e^{-\sqrt{s}}}{s}$ using complex inversion formula for $L_2$-transform leads to,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-\sqrt{s}}e^{st^2}}{s}\, ds,$$

using lemma 2.1, we get the following

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \{\frac{1}{\sqrt{\pi}}\int_0^{+\infty} \frac{e^{-(u+\frac{s}{4u})}}{\sqrt{u}}\, du\}\frac{e^{st^2}}{s}\, ds,$$

changing the order of integration, which is permissible, leads to

$$f(t) = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}}\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\frac{e^{-(\frac{1}{4u})s}}{s})e^{st^2}ds\}du,$$

but the value of the inner integral is $H(t^2 - \frac{1}{4u})$,hence,

$$f(t) = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}}\, H(t^2 - \frac{1}{4u})du,$$

using definition of Heaviside's function, one gets

$$f(t) = \frac{1}{\sqrt{\pi}}\int_{\frac{1}{4t^2}}^{+\infty} \frac{e^{-u}}{\sqrt{u}}\, du,$$

at this point, if we make a change of variable $u = w^2$ , we get finally

$$f(t) = \frac{2}{\sqrt{\pi}}\int_{\frac{1}{2t}}^{+\infty} e^{-w^2}dw = Erfc(\frac{1}{2t}). \qquad Q.E.D$$

**Example 2.2 :** Using complex inversion formula for $L_2$- transform show that

$$L_2^{-1}\{\frac{1}{\sqrt{s^2+1}}\} = \frac{2e^{-t^2}}{t\sqrt{\pi}}.$$

*Solution :* we have the following,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2e^{st^2}}{\sqrt{s+1}}\, ds,$$

at this point, in order to avoid the complex integration around a complicated key-hole contour, we use the integral representation for $\frac{1}{\sqrt{s+1}}$ as follows,

$$\frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-(s+1)u^2}du = \frac{1}{\sqrt{s+1}},$$

we get,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{\frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-(s+1)u^2}du\}\, e^{st^2}ds,$$

changing the order of integration, one has

$$f(t) = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}du\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-(t^2-u^2)s}ds\}$$

the value of inner integral, is $\delta(t^2-u^2)$ therefore,

$$f(t) = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}\delta(t^2-u^2)du,$$

letting $u^2 = x$ and using some elementary properties of Dirac-delta function, one has finally

$$f(t) = \frac{2e^{-t^2}}{t\sqrt{\pi}}.$$

# 3 Widder potential Transform

Widder potential Transform is defined as follows

$$\mathrm{P}\{f(t), y\} = \int_0^{\infty}\frac{tf(t)dt}{t^2+y^2}.$$

It is well-known that the second iterate of the $L_2$- transform is the Widder potential transform; [2] that is,

$$L_2^2\{f(t); y\} = L_2\{L_2\{f(t); u\}, y\} = \frac{1}{2}\mathrm{P}\{f(t), y\} = \frac{1}{2}\int_0^{+\infty}\frac{tf(t)}{t^2+y^2}\,dt.$$

**Lemma 3.1 :** We have the following relations,

- $$l_2\{\{L_2(f(x), y)\}, t\} = \frac{1}{2}P\{f(x), t\}.$$

- $$L_2P = PL_2.$$

*Proof:* see [2].

**Example 3.1 :**

$$\mathrm{P}\{\delta(t-a),y\}=\int_0^\infty \frac{t\delta(t-a)dt}{t^2+y^2}=\frac{a}{a^2+y^2},$$

on the other hand

$$L_2\{\{L_2(\delta(x-a),y)\},t\}=L_2\{ae^{-a^2y^2},t\}=\frac{a}{2(a^2+t^2)}=\frac{1}{2}P\{\delta(x-a),t\}$$

## 3.1 Complex inversion formula for Widder Potential Transform

**Theorem 3.1 :**Let

$$\mathrm{P}\{f(t),y\}=\int_0^\infty \frac{tf(t)dt}{t^2+y^2}=F(y),$$

then we have,

$$f(t)=\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}[\{\frac{2}{2\pi i}\int_{a-i\infty}^{a+i\infty}2F(\sqrt{y})e^{yw^2}dy\}_{w\to\sqrt{w}}]\ e^{wt^2}dw.$$

Provided that, all integrals involved are convergent.

*Proof:* That is the natural consequence of definition of Widder Potential transform and complex inversion formula for $L_2$- transform .

**Example 3.2:** Using complex inversion formula for Widder Potential transform show that

$$P^{-1}\{\frac{\pi}{2}e^{-my},t\}=\sin mt$$

*Solution :*

$$f(t)=\frac{1}{2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2[\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}2F(\sqrt{y})e^{yw^2}dy\}_{w\to\sqrt{w}}]\ e^{wt^2}dw$$

$$F(y)=\frac{\pi}{2}e^{-my}\Rightarrow 2F(\sqrt{y})=\pi\ e^{-m\sqrt{y}},$$

$$f(t)=\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}2[\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\pi\ e^{-m\sqrt{y}}e^{yw^2}dy\}_{w\to\sqrt{w}}]\ e^{wt^2}dw, \qquad (3\text{-}1)$$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{-m\sqrt{y}}e^{yw^2}dy=L^{-1}\{e^{-m\sqrt{y}};w^2\}=\frac{m}{2\sqrt{\pi w^6}}e^{-\frac{m^2}{4w^2}}.$$

Setting the above integral in relation (3-1), we get

$f(t) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} 2\pi \ \frac{m}{2\sqrt{\pi w^3}} e^{-\frac{m}{4w}} e^{wt^2} dw = \frac{m}{2}\sqrt{\pi}\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \ \frac{1}{\sqrt{w^3}} e^{-\frac{m^2}{4w}} e^{wt^2} dw\}.$

At this point, if we use relation (1-5) ,we obtain

$$\frac{m}{2}\sqrt{\pi}L^{-1}\{\frac{1}{\sqrt{w^3}}e^{-\frac{m^2}{4w}};t^2\} = \sin mt.$$

**Example 3.3 :** The following relations hold true,

1. $P\{\frac{\ln(x^2+a^2)}{x}, y\} = \frac{\pi}{y}\ln(a+y)$,
2. $P\{\frac{\ln x}{x}, y\} = \frac{\pi \ln y}{2y}$,
3. $P\{\frac{\exp(-\frac{x^2}{2})}{x}, y\} = \sqrt{\pi}\{\frac{e^{\frac{y^2}{2}}}{y}\}Erfc(\frac{y}{\sqrt{2}})$.

*Proof 1:*By definition of the Widder transform,we have the following relation

$$P\{\frac{\ln(x^2+a^2)}{x}, y\} = \int_0^\infty \frac{\ln(x^2+a^2)}{x}\frac{xdx}{x^2+y^2} = \int_0^\infty \frac{\ln(x^2+a^2)}{x^2+y^2}dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\ln(x^2+a^2)}{x^2+y^2}dx,$$

if we set

$$F(z) = \frac{\log(z+ai)}{z^2+y^2}, \quad z^2+y^2 = 0 \to z = \pm\, i\, y,$$

by residue theorem, we have

$$\oint F(z)dz = 2\pi i Res(F(z), z = iy),$$

or,

$$\int_{-R}^{+R}\frac{\log(x+ai)}{x^2+y^2}dx + \int_0^\pi \frac{\log(Re^{i\theta}+ai)}{R^2e^{2i\theta}+y^2}iRe^{i\theta}d\theta = 2\pi i Res(F(z), z = iy) = b_{-1},$$

$$b_{-1} = \lim_{z\to s}(z-s)F(z)$$
$$b_{-1} = \lim_{z\to iy}(z-iy)\frac{\log(z+ai)}{z^2+y^2} = \lim_{z\to iy}\frac{\log(z+ai)}{z+iy} = \frac{\log(y+a)i}{2iy}.$$

Letting $R \to \infty$

$$\int_{-\infty}^{+\infty}\frac{\log(x+ai)}{x^2+y^2}dx + \lim_{R\to\infty}\int_0^\pi \frac{\log(Re^{i\theta}+ai)}{R^2e^{2i\theta}+y^2}iRe^{i\theta}d\theta = 2\pi i\frac{\log(y+a)i}{2iy},$$

$$I_R = \int_0^\pi \frac{\log(Re^{i\theta}+ai)}{R^2e^{2i\theta}+y^2}iRe^{i\theta}d\theta,$$

we can show that $\lim_{R\to\infty} I_R = 0$ as follows,

$$
\begin{aligned}
0 \le |I_R| \ &\le \ \left|\int_0^{\pi} \frac{\log(Re^{i\theta}+ai)}{R^2e^{2i\theta}+y^2} iRe^{i\theta}d\theta\right| \\
&\le \ \int_0^{\pi} \left|\frac{\log(Re^{i\theta}+ai)}{R^2e^{2i\theta}+y^2} iRe^{i\theta}d\theta\right| \\
&\le \ \int_0^{\pi} \frac{R(\ln(R+a)+\pi)}{R^2-y^2} d\theta \to 0 \ \ as \quad R\to\infty \\
&\Rightarrow \int_0^{\infty} \frac{\ln(x^2+a^2)}{x^2+y^2} dx = \frac{\pi\ln(y+a)}{y}
\end{aligned}
$$

*Proof 2:* In relation(1) if we set $a=0$ , we get the desired result.

*Proof 3:* By definition of widder-transform , one has

$$P\{\frac{\exp(-\frac{x^2}{2})}{x}, y\} = \int_0^{+\infty} \frac{e^{-\frac{x^2}{2}}}{x^2+y^2}\ dx.$$

In order to calculate the above integral, we may use Fourier Cosine transform and Parseval's identity for cosine transforms as follows,
we have $F_c\{\exp(-ax), s\} = \frac{a\sqrt{2}}{(s^2+a^2)\sqrt{\pi}}$ and $F_c\{\exp(-\frac{x^2}{2}), s\} = \exp(-\frac{s^2}{2})$
upon using Parseval's identity, we get

$$\int_0^{+\infty} \{\frac{a\sqrt{2}}{(s^2+a^2)\sqrt{\pi}}\}\{\exp(-\frac{s^2}{2})\}ds = \int_0^{+\infty} \exp(-ax)\exp(-\frac{x^2}{2})dx,$$

after simplifying, we obtain

$$a\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{e^{-\frac{s^2}{2}}}{s^2+a^2}\ ds = \int_0^{+\infty} e^{-ax-\frac{x^2}{2}}dx,$$

or,

$$\int_0^{+\infty} \frac{e^{-\frac{s^2}{2}}}{s^2+a^2}\ ds = \frac{1}{a}\sqrt{\frac{\pi}{2}} \int_0^{+\infty} e^{-ax-\frac{x^2}{2}}dx,$$

finally,

$$\int_0^{+\infty} \frac{e^{-\frac{s^2}{2}}}{s^2+a^2}\ ds = \frac{\sqrt{\pi}\ e^{\frac{a^2}{2}}}{a}\ Erfc(\frac{a}{\sqrt{2}}).$$

## 3.2 Application of Widder Potential transform in evaluating the Integrals

**Example 3.4:** Show that

$$\int_0^{+\infty} x^{2\lambda-1}\sin tx\ dx = t^{-2\lambda}\Gamma(2\lambda)\sin\pi\lambda.$$

*Solution :* Let us assume that $I(t)=\int_0^{+\infty}\ x^{2\lambda-1}\sin tx\ dx$ , taking Widder transform of the two sides, yields

$$P\{I(t),y\}=\int_0^{+\infty}\frac{tdt}{t^2+y^2}(\int_0^{+\infty}\ x^{2\lambda-1}\sin tx\ dx),$$

changing the order of integration and using Example 3.2, we get

$$P\{I(t),y\}=\int_0^{+\infty}\ x^{2\lambda-1}dx(\int_0^{+\infty}\ \frac{t\sin xt}{t^2+y^2}dt)=\frac{\pi}{2}\int_0^{+\infty}\ x^{2\lambda-1}e^{-yx}dx,$$

or,

$$P\{I(t),y\}=\frac{\pi}{2}\{\frac{\Gamma(2\lambda)}{y^{2\lambda}}\},$$

at this point, we use complex inversion formula for Widder potential transform to get

$$I(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2[\{\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}2\{\frac{\Gamma(2\lambda)}{\pi y^{\lambda}}e^{yw^2}dy\}_{w\to\sqrt{w}}]\ e^{wt^2}dw,$$

hence,

$$I(t)=t^{-2\lambda}\Gamma(2\lambda)\sin\pi\lambda.$$

## 3.3 $E_{2,1}$-Transform

$E_{2,1}$-transform is defined as follows

$$E_{2,1}\{f(t),y\}=\int_0^{\infty}x\exp(x^2y^2)E_1(x^2y^2)f(x)dx,$$

where $E_1(x)$ is the exponential integral defined as

$$E_1(x)=\int_x^{+\infty}\frac{\exp(-\xi)}{\xi}\,d\xi,$$

It is well-known that the third iterate of the $L_2$-transform is the $E_{2,1}$-transform [2] , that is,

$$L_2^3\{f(t);y\}=L_2\{L_2\{L_2\{f(t);u\},v\},y)=\frac{1}{4}E_{2,1}\{f(t),y\}=F(y). \qquad (3\text{-}2)$$

**Example 3.5 :** Show that

$$E_{2,1}\{\delta(t-a),y\}=ae^{a^2y^2}E_1(a^2y^2).$$

*Proof:* We know that, $L_2\{\{L_2(f(x),y)\},t\}=\frac{1}{2}P\{f(x),t\}$

$$L_2^3\{f(t);y\} = L_2\{L_2\{L_2\{f(t);u\},v\},y) = \frac{1}{4}E_{2,1}\{f(t),y\} = F(y),$$

on the other hand we have

$$\mathrm{P}\{\delta(t-a),y\} = \int_0^\infty \frac{t\delta(t-a)dt}{t^2+y^2} = \frac{a}{a^2+y^2},$$

finally we have

$$\begin{aligned}
E_{2,1}\{\delta(t-a),y\} &= 4L_2^3\{\delta(t-a);y\} = 4\times\frac{1}{2}L_2\{P\{\delta(t-a),u\},y\} \\
&= 2L_2\{\frac{a}{a^2+u^2},y\} = 2\times\frac{1}{2}L\{f(\sqrt{u});y^2\} \\
&= L\{\frac{a}{a^2+u},y^2\} = aL\{\frac{1}{u+a^2};y^2\} \\
&= ae^{a^2y^2}E_1(a^2y^2)
\end{aligned}$$

# 4 Conclusion

In this article, authors provide some identities involving new transforms such as Widder Potential and $E_{2,1}$-transform. They also derive complex inversion formula for the above-mentioned transforms .They conclude by remarking that many identities involving various integral transforms can be obtained and some other infinite integrals can be evaluated by applying the results considered here.

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