

Complex Inversion Formula for Exponential Integral Transform with Applications

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Abstract

In this article, we derive a complex inversion formula and some new theorems related to Stieltjes-transform and exponential integral transform defined in [2],[3],we also give some illustrative examples.

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1 Introduction

Integral transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way. Laplace transforms provides a powerful method for solving differential and integral equations. Recently, many other Transforms have been developed, but most have limited applicability. This work also contains some discussion on several other integral transforms that have been used recently successfully in the solution of certain boundary value problems and in other applications. Included in this category are, Stieltjes-transform, exponential integral transform ([1],[2],[3]) which have also been given. The Laplace-transform of the function is defined as

$$L\{f(t)\} = F(s) = \int_0^{+\infty} \exp(-st)f(t)dt. \quad (1-1)$$

The inverse laplace-transform is given by the integral relationship

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds, \quad (1-2)$$

where σ is any suitably chosen large positive constant. This improper integral is a contour integral taken along the vertical line $s = \sigma + i\tau$ in complex

s-plane. for more detail see[4].

Example 1.1 : The following are immediate consequences of definition (1-1)

1.

$$L\{\delta(x - a)\} = \exp(-as).$$

2.

$$L\{\exp(-ax)\} = \frac{1}{s + a}.$$

3.

$$L\left\{\frac{1}{x + a}\right\} = e^{-as} E_1(as).$$

Example 2.1: Using complex inversion formula for Laplace transform to find

$$f(t) = L^{-1}\left\{\frac{\sqrt{s+1}}{s}\right\}.$$

Solution : We have the following relation,

$$\frac{\sqrt{s+1}}{s} = \frac{1}{\sqrt{s+1}} + \frac{1}{s\sqrt{s+1}},$$

therefore, it is sufficient to evaluate $L^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\}$, using relation (1-2), leads to

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{\sqrt{s+1}} ds,$$

at this point, in order to avoid the complex integration around a complicated key-hole contour, we use the integral representation for $\frac{1}{\sqrt{s+1}}$ as follows,

$$\frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-(s+1)u^2} du = \frac{1}{\sqrt{s+1}}.$$

Inserting in the above relationship, we get,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-(s+1)u^2} du \right\} e^{st} ds,$$

changing the order of integration, to obtain

$$f(t) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-u^2} du \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{(t-u^2)s} ds \right\},$$

but, the value of inner integral, is $\delta(t - u^2)$ therefore,

$$f(t) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-u^2} \delta(t - u^2) du,$$

letting $u^2 = x$ and using some elementary properties of delta function, one has finally

$$L^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} = f(t) = \frac{e^{-t}}{\sqrt{\pi t}},$$

and,

$$L^{-1}\left\{\frac{\sqrt{s+1}}{s}\right\} = L^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} + L^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\},$$

hence,

$$L^{-1}\left\{\frac{\sqrt{s+1}}{s}\right\} = \frac{e^{-t}}{\sqrt{\pi t}} + \int_0^t H(t-x) \frac{e^{-x}}{\sqrt{\pi x}} dx,$$

finally,

$$L^{-1}\left\{\frac{\sqrt{s+1}}{s}\right\} = \frac{e^{-t}}{\sqrt{\pi t}} + \operatorname{erf}(\sqrt{t}).$$

2 Stieltjes Transform

Stieltjes Transform is defined as follows

$$\wp\{f(t), y\} = \int_0^\infty \frac{f(t)dt}{t+y}.$$

It is well-known that the second iterate of the Laplace transform is the Stieltjes-transform [2],[3], that is,

$$L^2\{f(t); y\} = L\{L\{f(t); u\}, y\} = \wp\{f(t), y\} = F(y). \quad (2-1)$$

Example 2.1: For $f(t) = \delta(t-a)$, we can verify the above relationship we have

$$\wp\{\delta(t-a), y\} = \int_0^\infty \frac{\delta(t-a)dt}{t+y} = \frac{1}{a+y},$$

on the other hand, second iterate of the Laplace transform of $\delta(t-a)$ is,

$$L\{\delta(t-a), u\} = e^{-au},$$

and

$$L\{e^{-au}, y\} = \frac{1}{y+a}. \quad Q.E.D$$

Example 2.2: show that

$$\wp\{t^{\mu-1}, y\} = \frac{\pi y^{\mu-1}}{\sin \mu\pi} \quad (0 < \mu < 1).$$

Solution : Upon using relationship(3-1)

$$\begin{aligned}
L^2\{f(t); y\} &= L\{L\{f(t); u\}, y\} = \wp\{f(t), y\} = \int_0^{+\infty} \frac{t^{\mu-1}}{t+y} dt \\
&= L^2\{f(t); y\} = L\{L\{f(t); u\}, y\} = L\{L\{t^{\mu-1}, u\}, y\} = L\left\{\frac{\Gamma(\mu)}{u^\mu}, y\right\} \\
&= \Gamma(\mu)L\{u^{-\mu}, y\} = \Gamma(\mu)\Gamma(1-\mu)y^{\mu-1} = \frac{\pi}{\sin \mu\pi}y^{\mu-1} \quad (0 < \mu < 1).
\end{aligned}$$

2.1 Complex inversion formula for Stieltjes Transform

The inverse Stieltjes Transform is given by the following integral relationship

$$f(t) = \wp^{-1}\{F(y)\} = \frac{1}{(2\pi i)^2} \int_{\lambda-i\infty}^{\lambda+i\infty} \left(\int_{\sigma-i\infty}^{\sigma+i\infty} F(y)e^{yx} dy \right) e^{xt} dx, \quad (2-2)$$

where σ, λ are any suitably chosen large positive constants. This improper integral is a contour integral taken along the vertical lines $y = \sigma + i\tau$ and $x = \lambda + i\eta$ in complex-plane.

Example 2.3: Show that

$$\wp^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, t\} = t^{-\frac{3}{4}}.$$

Proof: Since Stieltjes transform is second iterate of Laplace transform, we can re-write relation (2-2) as follows

$$f(t) = L^{-1}\{L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\}, t\},$$

using the fact that, $t^\nu = L^{-1}\left\{\frac{\Gamma(\nu+1)}{s^{\nu+1}}\right\}$ hence,

$$L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\} = \frac{\pi \sqrt{2}}{\Gamma(\frac{3}{4})} x^{-\frac{1}{4}},$$

finally,

$$f(t) = L^{-1}\{L^{-1}\{\pi \sqrt{2} y^{-\frac{3}{4}}, x\}, t\} = L^{-1}\left\{\frac{\pi \sqrt{2}}{\Gamma(\frac{3}{4})} x^{-\frac{1}{4}}, t\right\} = \frac{\pi \sqrt{2}}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} t^{-\frac{3}{4}} = t^{-\frac{3}{4}}.$$

Example 2.4: Evaluate $\wp\{\cos t\}$ and $\wp\{\sin t\}$.

Solution: Let us assume that Γ_R be the closed contour composed of a quarter circle $z = Re^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$ in first quadrant and two line segments $[0, R], [0, iR]$ traversed in counter clockwise, and let $F(z) = \frac{e^{iz}}{z+a}$, $a > 0$, then we have

$$\oint_{\Gamma_R} \frac{e^{iz}}{z+a} dz = 0$$

now, we expand the above integral to obtain

$$\int_0^R \frac{e^{ix}}{x+a} dx + \int_0^{\frac{\pi}{2}} \frac{e^{(iRi\theta)}}{Re^{i\theta}+a} iRe^{i\theta} d\theta + \int_R^0 \frac{e^{i(iy)}}{iy+a} idy = 0,$$

taking limit as R tends to infinity, to get

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx + \lim_{R \rightarrow +\infty} \int_0^{\frac{\pi}{2}} \frac{e^{(iRi\theta)}}{Re^{i\theta}+a} iRe^{i\theta} d\theta + \int_{+\infty}^0 \frac{e^{i(iy)}}{iy+a} idy = 0,$$

the second integral tends to zero (in absolute value) therefore, one has the following

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx + \int_{+\infty}^0 \frac{e^{i(iy)}}{iy+a} idy = 0,$$

or,

$$\int_0^{+\infty} \frac{e^{ix}}{x+a} dx = \int_0^{+\infty} \frac{ye^{-y}dy}{y^2+a^2} + i \int_0^{+\infty} \frac{ae^{-y}}{y^2+a^2} dy.$$

Separating real and imaginary parts of both sides of the above relationship, to get

$$\begin{aligned} \wp(\cos x) &= \int_0^{+\infty} \frac{\cos x}{x+a} dx = \int_0^{+\infty} \frac{xe^{-x}}{x^2+a^2} dx, \\ \wp(\sin x) &= \int_0^{+\infty} \frac{\sin x}{x+a} dx = \int_0^{+\infty} \frac{ae^{-x}}{x^2+a^2} dx. \end{aligned}$$

3 Ξ_1 -Transform

E_1 -Transform is defined as follows

$$\Xi_1\{f(t), y\} = \int_0^{\infty} \exp(xy) E_1(xy) f(x) dx$$

where $E_1(x)$ is the exponential integral defined as

$$E_1(x) = \int_x^{+\infty} \frac{\exp(-\xi)}{\xi} d\xi$$

It is well-known that the third iterate of the Laplace transform is the Ξ_1 -Transform [2],[3], that is,

$$L^3\{f(t); y\} = L\{L\{L\{f(t); u\}, v\}, y\} = \Xi_1\{f(t), y\} = F(y). \quad (3-1)$$

Example 3.1 : The following relations hold true

1.

$$\Xi_1\{\delta(x-a)\} = e^{-as} E_1(as).$$

2.

$$\Xi_1(x^\lambda, y) = \frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)} y^{\lambda-1}.$$

Theorem 3.1 : The following identities hold true,

1.

$$\wp\{L\{g(x); s+u\}; y\} = L\{e^{xy} E_1(xy)g(x); s\}.$$

2.

$$\wp\{L\{g(x); s+u\}; y\} = \frac{1}{y} L\{E_1(x)g\left(\frac{x}{y}\right); \frac{s}{y} - 1\}.$$

3.

$$\wp\left\{\frac{e^{-ax}}{\sqrt{\pi x}}, y\right\} = \sqrt{\frac{\pi}{y}} \exp(ay) \operatorname{Erfc}(\sqrt{ay}).$$

4.

$$\wp\left\{\sqrt{\frac{k}{\pi}} x^{-\frac{2}{3}} \exp\left(-\frac{k}{x}\right), y\right\} = \frac{1}{2y} - \sqrt{k\pi} \exp\left(\frac{k}{y}\right) y^{-\frac{3}{2}} \operatorname{Erf}\left(\frac{\sqrt{k}}{y}\right).$$

5.

$$\wp\{E_1(x), y\} = L\left\{\frac{\ln(x+1)}{x}; y\right\}.$$

6.

$$\Xi_1\{f(x); y\} = \int_0^\infty e^{-x(-y)} E_1(xy) f(x) dx = L\{E_1(xy) f(x); -y\}.$$

7.

$$\wp\{x \exp(-ax), y\} = \frac{1}{a} - y e^{ay} E_1(ay).$$

Proof 1 :

$$\begin{aligned} f(x) &= e^{-sx} g(x) \Rightarrow L\{f(x); u\} = L\{e^{-sx} g(x); u\} = \int_0^{+\infty} e^{-sx} e^{-ux} g(x) dx \\ &= \int_0^{+\infty} e^{-x(s+u)} g(x) dx = L\{g(x); s+u\} \Rightarrow \end{aligned}$$

$$\wp\{L\{f(x); u\}; y\} = \wp\{L\{g(x); s+u\}; y\}$$

$$\begin{aligned} &\Rightarrow \wp\{L\{f(x); u\}; y\} = \int_0^{+\infty} \frac{L\{e^{-sx} g(x); u\}}{u+y} du \\ &= \int_0^{+\infty} \frac{1}{u+y} \left\{ \int_0^{+\infty} e^{-ux} e^{-sx} g(x) dx \right\} du \end{aligned}$$

If we change the order of integration and letting $y+u=z$ in the inner integral, we get,

$$\int_0^\infty e^{-sx} g(x) dx \int_0^\infty \frac{e^{-ux}}{y+u} du = \int_0^\infty e^{-sx} g(x) dx \int_y^\infty \frac{e^{-(z-y)x}}{z} dz,$$

or,

$$\int_0^\infty e^{-sx} g(x) dx e^{yx} \int_y^\infty \frac{e^{-zx}}{z} dz,$$

making a new change of variable $xz = t$, one has

$$\int_0^\infty e^{-sx} g(x) dx \left\{ e^{xy} \int_{xy}^\infty \frac{e^{-t}}{t} dt \right\} = \int_0^\infty e^{-sx} g(x) e^{xy} E_1(xy) dx = L\{e^{xy} E_1(xy) g(x); s\}.$$

Proof 2 : Setting $xy = t$, in part 1 of Theorem 3.1 to get,

$$\wp\{L\{g(x); s+u\}; y\} = \int_0^\infty e^{-s\frac{t}{y}} E_1(t) g\left(\frac{t}{y}\right) e^t \frac{1}{y} dt = \frac{1}{y} \int_0^\infty e^{-x\left(\frac{s}{y}-1\right)} E_1(x) g\left(\frac{x}{y}\right) dx,$$

or finally,

$$= \frac{1}{y} L\{E_1(x) g\left(\frac{x}{y}\right); \frac{s}{y} - 1\}.$$

Proof 3:

$$\wp\left\{\frac{e^{-ax}}{\sqrt{\pi x}}; y\right\} = L\left\{L\left\{\frac{e^{-ax}}{\sqrt{\pi x}}; z\right\}; y\right\} = L\left\{\int_0^\infty \frac{e^{-zx} e^{-ax}}{\sqrt{\pi}} x^{-\frac{1}{2}} dx; y\right\} = L\left\{L\left\{\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}}; z+a\right\}; y\right\} =$$

$$L\left\{\frac{1}{(z+a)^{\frac{1}{2}}}; y\right\} = \int_0^\infty e^{-zy} (z+a)^{-\frac{1}{2}} dz,$$

Let $z+a = \frac{t^2}{y}$ to get

$$L\left\{\frac{1}{(z+a)^{\frac{1}{2}}}; y\right\} = e^{ay} \frac{2}{\sqrt{y}} \int_{\sqrt{ay}}^\infty e^{-t^2} dt = e^{ay} \sqrt{\frac{\pi}{y}} \operatorname{erfc}(\sqrt{ay}).$$

Proof 5:

$$\wp\{E_1(x)\} = \int_0^\infty \frac{1}{x+y} \left\{ \int_x^\infty \frac{e^{-u}}{u} du \right\} dx = \int_0^\infty \frac{e^{-u}}{u} \left\{ \int_0^u \frac{1}{x+y} dx \right\} du = \int_0^\infty \frac{e^{-u}}{u} \left(\ln \frac{u+y}{y} \right) du \Rightarrow$$

At this point, we set $\frac{u+y}{y} - 1 = x$

$$\int_0^\infty e^{-yx} \frac{\ln(x+1)}{x} dx = L\left\{\frac{\ln(x+1)}{x}; y\right\}.$$

Proof 6:

$$E_1\{f(x); y\} = \int_0^\infty e^{-x(-y)} E_1(xy) f(x) dx = L\{E_1(xy) f(x); -y\}.$$

Proof 7:

$$\wp\{x \exp(-ax), y\} = \int_0^\infty \frac{x e^{-ax}}{x+y} dx,$$

making a change of variable $x + y = \frac{u}{a}$, one has

$$\frac{1}{a} - y e^{ay} E_1(ay) = \frac{1}{a} - y e^{ay} E_1(ay).$$

3.1 Complex inversion formula for Ξ_1 -Transform

The inverse Ξ_1 - Transform is given by the following integral relationship

$$f(t) = \Xi_1^{-1}\{F(y)\} = \frac{1}{(2\pi i)^3} \int_{\mu-i\infty}^{\mu+i\infty} \left(\int_{\lambda-i\infty}^{\lambda+i\infty} \left(\int_{\sigma-i\infty}^{\sigma+i\infty} F(y) e^{yx} dy \right) e^{xz} dx \right) e^{zt} dz \quad (3-2)$$

where σ , λ , μ are any suitably chosen large positive constants.

This improper integral is a contour integral taken along the vertical lines $y = \sigma + i\tau$, $x = \lambda + i\eta$ and $z = \mu + i\zeta$ in complex - plane.

Example 3.2 : Show that

$$\Xi_1^{-1}\left\{\frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)} y^{\lambda-1}; u\right\} = x^{-\lambda}.$$

Solution:

$$\begin{aligned} L^{-1}\{L^{-1}\{L^{-1}\left\{\frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)} y^{\lambda-1}; u\right\}; v\}x\} &= L^{-1}\{L^{-1}\left\{\frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)\Gamma(1-\lambda)} u^{-\lambda}; v\right\}; x\} \\ &= L^{-1}\left\{\frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)\Gamma(1-\lambda)\Gamma(\lambda)} v^{\lambda-1}; x\right\} = \frac{\pi \Gamma(1-\lambda)}{\sin(\lambda\pi)\Gamma(1-\lambda)\Gamma(\lambda)\Gamma(1-\lambda)} x^{-\lambda} = x^{-\lambda}. \end{aligned}$$

Example 3.3 : Show that

$$\int_0^\infty L\{f(x); y\} \wp\{g(u); y\} dy = \int_0^\infty f(x) \Xi_1\{g(u); x\} dx.$$

Solution:

$$\begin{aligned} \int_0^\infty L\{f(x); y\} \wp\{g(u); y\} dy &= \int_0^\infty \wp\{g(u); y\} \left[\int_0^\infty e^{-xy} f(x) dx \right] dy = \\ \int_0^\infty f(x) \left[\int_0^\infty e^{-xy} \wp\{g(u); y\} dy \right] dx &= \int_0^\infty f(x) L\{\wp\{g(u); y\}; x\} dx = \int_0^\infty f(x) \Xi_1\{g(u); x\} dx. \end{aligned}$$

Example 3.4 :

$$\int_0^\infty E_1\{f(x); y\} \wp\{g(u); y\} dy = \int_0^\infty g(u) \wp\{E_1\{f(x); y\}; u\} du.$$

Solution:

$$\begin{aligned} \int_0^\infty E_1\{f(x); y\} \wp\{g(u); y\} dy &= \int_0^\infty E_1\{f(x); y\} \left[\int_0^\infty \frac{g(u)}{u+y} du \right] dy = \\ \int_0^\infty g(u) \left[\int_0^\infty \frac{E_1\{f(x); y\}}{u+y} dy \right] du &= \int_0^\infty g(u) \wp\{E_1\{f(x); y\}; u\} du. \end{aligned}$$

Example 3.5 :

$$\int_0^\infty E_1\{f(x); y\} \wp\{g(u); y\} dy = \int_0^\infty \wp\{f(x); y\} E_1\{g(u); y\} dy.$$

Solution:

$$\begin{aligned} \int_0^\infty E_1\{f(x); y\} \wp\{g(u); y\} dy &= \int_0^\infty L\{\wp\{f(x); v\}; y\} \wp\{g(u); y\} dy = \\ \int_0^\infty \wp\{g(u); y\} \left[\int_0^\infty e^{-yv} \wp\{f(x); v\} dv \right] dy, \\ \int_0^\infty \wp\{f(x); v\} \left[\int_0^\infty e^{-yv} \wp\{g(u); y\} dy \right] dv &= \int_0^\infty \wp\{f(x); v\} L\{\wp\{g(u); y\}; v\} dv = \\ \int_0^\infty \wp\{f(x); v\} E_1\{g(u); v\} dv &= \int_0^\infty \wp\{f(x); y\} E_1\{g(u); y\} dy. \end{aligned}$$

Example 3.6 : Show that

$$\frac{\sin \lambda \pi}{\pi \Gamma(\lambda)} \int_0^\infty L\left\{\frac{1}{u+a}; y\right\} y^{\lambda-1} dy = a^{-\lambda}.$$

Solution: We have from Example 3.5

$$\begin{aligned} \frac{\sin \lambda \pi}{\pi \Gamma(\lambda)} \int_0^\infty L\left\{\frac{1}{u+a}; y\right\} y^{\lambda-1} dy &= \frac{\sin \lambda \pi}{\pi \Gamma(\lambda)} \int_0^\infty L\{\wp\{\delta(x-a); u\}; y\} P\left\{\frac{x^{\lambda-1}}{\Gamma(1-\lambda)\Gamma(\lambda)}; y\right\} dy \\ &= \frac{\sin^2 \lambda \pi}{\pi^2 \Gamma(\lambda)} \int_0^\infty E_1\{\delta(x-a); y\} \wp\{x^{\lambda-1}; y\} \xrightarrow{Ex5.5} \frac{\sin^2 \lambda \pi}{\pi^2 \Gamma(\lambda)} \int_0^\infty \wp\{\delta(x-a); y\} E_1\{x^{\lambda-1}; y\} dy \\ &= \frac{\sin \lambda \pi}{\pi} \int_0^\infty \frac{y^{-\lambda}}{y+a} dy = \frac{\sin \lambda \pi}{\pi} \wp\{y^{-\lambda}; a\} = \frac{\sin \lambda \pi}{\pi} \Gamma(\lambda) \Gamma(1-\lambda) a^{-\lambda} = a^{-\lambda}. \end{aligned}$$

4 Conclusion

In this article, authors provide complex inversion formula for Stieltjes - transform and Exponential integral transform. Some identities involving these new transforms are given. Our primary objective is to introduce methods to use the integral transform rather than concerning ourselves too deeply with the general theorems. We have included a large number of worked examples to illustrate the applicability of these new transforms. We conclude by remarking that many identities involving various integral transforms can be obtained and some other infinite integrals can be evaluated by applying the results considered here.

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