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He's Homotopy Perturbation Method for Solving Helmholtz Equation

J. Biazar[1]

Department of Mathematics, Faculty of Sciences
The University of Guilan, P.O. Box 1914, Rasht, Iran
biazar@guilan.ac.ir

F. Azimi

Payamnoor University, Fariman-Branch, Iran
30019@yahoo.com

Abstract

In this paper He's homotopy perturbation method (HPM) is applied to solve the Helmholtz equation.To illustrate the method some examples are provided. The results reveal that this method is very effective and simple.

Keywords: Homotopy perturbation method, Helmholtz equation

1 Introduction

Homotopy perturbation method (HPM), proposed first by He [1,2], for solving differential and integral equations. The method which is a coupling of the traditional perturbation method and homotopy in topology deforms continuously the problem in hand to a simple one which can be easily solved. In contrast to the traditional perturbation methods, the proposed method does not require a small parameter in the equation. In this method a homotopy with an embedding parameter $p \in [0, 1]$ as a small parameter is constructed.

Partial differential equations can describe many physical phenomena in different fields of science, and engineering. These linear and nonlinear models play important roles in applied science. In this paper we implement homotopy perturbation method for finding the exact solution of the Helmholtz equations.

[1]Corresponding author

This equation appear in such divers phenomena as: elastic waves in solids including vibrating string, bars, memberanes, sound or ecoustic, electromagnetic waves, and nuclear reactors [5,6].

Consider Helmholtz equation on a given region $\Re$ in the xy-plane in the following form:

$$\bigtriangledown^2 u + f(x,y)u = g(x,y). \qquad (1)$$

Where $u(x,y)$ is known on the boundary of $\Re$. The boundary and initial condition could be given by the following functions:

$$u(0,y) = \psi_1(y), \qquad u_x(0,y) = \psi_2(y), \qquad (2)$$

$$u(x,0) = \psi_3(y), \qquad u_y(x,0) = \psi_4(y). \qquad (3)$$

Where $\psi_1(y),\ \psi_2(y),\ \psi_3(y)$ and $\psi_4(y)$are known functions [3].

2 Homotopy Perturbation Method for Hemholtz equation

Homotopy perturbation method will addressed in [1,2] can be applied to equation (1) to give the following equation:

$$H(v,p) = (1-p)v_{xx} + p\ [v_{xx} + v_{yy} + f(x,y)v - g(x,y)] = 0. \qquad (4)$$

Which is equivalent to

$$H(v,p) = v_{xx} + p\ [v_{yy} + f(x,y)v - g(x,y)] = 0. \qquad (5)$$

Where

$$v(r,p) : \Omega \times [0,1] \longrightarrow \Re. \qquad (6)$$

In Eq.(5), $p \in [0,1]$ is the mentioned small embedding parameter.
In this method the solution of Eq.(5) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \qquad (7)$$

Then by substitution the series into the Eq.(4) or Eq.(5) and equating the coefficient of the terms by the same power in p, we derive a successive procedure to determine terms, $v_i's$.
In theory the solution will be the limit of v as p tends to unity. But in praction if the series could not be recognized as the series of a known function, first few terms will be taken as an approximation to the solution $u = \sum_{i=1}^{n} v_i$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots \qquad (8)$$

3 Numerical Examples

In order to assess the accuracy of HPM, and to illustrate the method in more details, we consider the following three examples.

Example 1. Consider the Helmholtz equation as follows [4].

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0. \tag{9}$$

With the initial conditions

$$u(0,y) = y, \qquad u_x(0,y) = y + cosh(y). \tag{10}$$

In order to solve Eq.(9), using HPM, we construct the following homotopy for this equation:

$$v_{xx} + p\,(v_{yy} - v) = 0. \tag{11}$$

Substituting v from Eq.(7) into Eq.(11) and equating the terms with identical powers of p, we have:

$$p^0 \;:\; \frac{\partial^2 v_0}{\partial x^2} = 0, \quad v_0(0,y) = y, \quad (v_0)_x(0,y) = y + cosh(y), \tag{12}$$

$$p^1 \;:\; \frac{\partial^2 v_1}{\partial x^2} = -\frac{\partial^2 v_0}{\partial y^2} + v_0 \quad , v_1(0,y) = 0, \quad (v_1)_x(0,y) = 0, \tag{13}$$

$$p^2 \;:\; \frac{\partial^2 v_2}{\partial x^2} = -\frac{\partial^2 v_1}{\partial y^2} + v_1, \quad v_2(0,y) = 0, \quad (v_2)_x(0,y) = 0, \tag{14}$$

$$\vdots$$

Solving this system, recursively, we derive the following results:

$$v_0(x,y) = (y + cosh\;\, y)x + y,$$

$$v_1(x,y) = \tfrac{1}{6}yx^3 + \tfrac{1}{2}yx^2,$$

$$v_2(x,y) = \tfrac{1}{120}yx^5 + \tfrac{1}{24}yx^4,$$

$$v_3(x,y) = \tfrac{1}{5040}yx^7 + \tfrac{1}{720}yx^6,$$

$$\vdots$$

And so,

$$\begin{aligned} u(x,y) &= (y+cosh\ y)x + y + \frac{yx^3}{3!} + \frac{yx^2}{2!} + \frac{yx^5}{5!} + \frac{yx^4}{4!} + \ldots \\ &= xcoshy + y(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots) \\ &= xcoshy + yExp\ (x). \end{aligned} \quad (15)$$

Which is an exact solution and is exactly the same as those obtained by Adomian decomposition method [4].

Example 2. Consider another form of Helmholtz equation, with indicated initial conditions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = xsiny, \quad (16)$$

$$u(0,y) = 1, \qquad u_x(0,y) = 0. \quad (17)$$

Now construct a homotopy which satisfies the following relation:

$$v_{xx} + p\ (v_{yy} + v - xsiny) = 0. \quad (18)$$

With substituting v from Eq.(7) into Eq.(18), and equating the coefficients of terms with the same powers of p, we obtain:

$$\begin{aligned} p^0 &: \frac{\partial^2 v_0}{\partial x^2} = 0, \quad v_0(0,y) = 1, \quad (v_0)_x(0,y) = 0, && (19) \\ p^1 &: \frac{\partial^2 v_2}{\partial x^2} = -\frac{\partial^2 v_0}{\partial y^2} - v_0 + xsiny, \quad v_1(0,y) = 0, \quad (v_1)_x(0,y) = 0, && (20) \\ p^2 &: \frac{\partial^2 v_2}{\partial x^2} = -\frac{\partial^2 v_1}{\partial y^2} - v_1, \quad v_2(0,y) = 0, \quad (v_2)_x(0,y) = 0, && (21) \\ p^3 &: \frac{\partial^2 v_3}{\partial x^2} = -\frac{\partial^2 v_2}{\partial y^2} - v_2, \quad v_3(0,y) = 0, \quad (v_3)_x(0,y) = 0, && (22) \\ & \vdots \end{aligned}$$

Solving this system, we obtain the following solutions for v_0, v_1, v_2 and v_3, etc.

$v_0(x,y) = 1,$

$v_1(x,y) = -\frac{x^2}{2!} + \frac{x^3}{3!}siny,$

$v_2(x,y) = \frac{x^4}{4!}$,

$v_3(x,y) = -\frac{x^6}{6!}$

$$\vdots$$

And the solution will be written as,

$$\begin{aligned} u(x,y) &= \frac{x^3}{6} siny + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots . \\ &= \frac{x^3}{3!} siny + cosx. \end{aligned} \tag{23}$$

This is an exact solution.

Example 3. Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + xu = 2 + x^3 + xy, \quad 0 \le x \le 1. \tag{24}$$

With the initial conditions

$$u(0,y) = y, \quad u_x(0,y) = 0. \tag{25}$$

We construct a homotopy which satisfies the following relation:

$$v_{xx} + p\,[v_{yy} + xv - (2 + x^3 + xy)] = 0. \tag{26}$$

Substituting v from Eq.(7) into Eq.(26) and equating the terms with identical powers of p , we have:

$$p^0 \;:\; \frac{\partial^2 v_0}{\partial x^2} = 0, \quad v_0(0,y) = y, \quad (v_0)_x(0,y) = 0, \tag{27}$$

$$p^1 \;:\; \frac{\partial^2 v_1}{\partial x^2} = -\frac{\partial^2 v_0}{\partial y^2} + xv_0 - 2 - x^3 - xy, \quad v_1(0,y) = 0, \quad (v_1)_x(0,y) = 0, \tag{28}$$

$$p^2 \;:\; \frac{\partial^2 v_2}{\partial x^2} = -\frac{\partial^2 v_1}{\partial y^2} + xv_1, \quad v_2(0,y) = 0, \quad (v_2)_x(0,y) = 0, \tag{29}$$

$$p^3 \;:\; \frac{\partial^2 v_3}{\partial x^2} = -\frac{\partial^2 v_2}{\partial y^2} + xv_2, \quad v_3(0,y) = 0, \quad (v_3)_x(0,y) = 0, \tag{30}$$

$$\vdots$$

Solving this system we obtain:
$v_0(x,y) = y$,

$v_1(x,y) = x^2 + \frac{x^5}{20}$,

$v_2(x,y) = -\frac{x^8}{1120} - \frac{x^5}{20}$,

$v_3(x,y) = -\frac{x^{11}}{123200} + \frac{x^8}{1120}$,

$\vdots$

$$\begin{aligned} u(x,y) &= v_0 + v_1 + v_2 + v_3 + \dots \\ &= y + x^2. \end{aligned} \quad (31)$$

4 Conclusion

In this letter, we have successfully developed HPM to obtain exact solution of Helmholtz equation. It is apparently seen that this method is very powerful and efficient technique. The obtained results demonstrate the reliability of the algorithm and its applicability to some partial differential equations.

References

[1] J. H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999), 257-262.

[2] J. H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-linear Mech., 35 (2000), 37-43.

[3] Salah M. El-Sayed, Dogan Kaya, Comparing numerical methods for Helmholtz equation model problem, Appl. Math. Comput., 150 (2004), 763-773.

[4] Shaher Monani, Application of He's variational iteration method to Helmholtz equation, Chaos, Solition and Fractals, 27 (2006), 1119-1123.

[5] R. L. Burden, J. D. Faires, Numerical Analysis, PWS Publishing Company, Boston, 1993.

[6] C. F. Gerald, P. O. Wheatly, Applied Numerical Analysis, Addison Wesley, California, 1994.

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