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Exact Solutions of Some Hyperbolic Equations with Initial Conditions

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Abstract

Adomian decomposition method has been applied to solve many functional equations so far. In this article, we have used this method to solve some hyperbolic equations, Cuachy problems, with initial conditions.

Keywords: Adomian decomposition method; Hyperbolic equations; Cauchy problems

1 Introduction

Analytical methods commonly used for solving hyperbolic equations are very restricted and can be used in very special cases.
Adomian decomposition method has a useful feature in that it provides the solution in a rapid convergent power series with elegantly computable convergence of the solution.
The decomposition method has proven to be very effective and results in considerable savings in computation time.
In this work we focus our study to hyperbolic equations.
These equations can be written in the form

$$a_{11}u_{tt} + 2a_{12}u_{tx} + a_{22}u_{xx} + b_1u_t + b_2u_x + cu = f \quad where \qquad a_{12}^2 - a_{11}a_{22} > 0 \tag{1}$$

with initial condintions:

$$u(x,0) = f_0(x)$$
$$\frac{\partial u(x,0)}{\partial t} = f_1(x) \tag{2}$$

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2 Analysis of Adomian decomposition method

In an operator Form, Equation.(1) becomes

$$L(u)=-\frac{1}{a_{11}}(2a_{11}u_{tx}+a_{22}u_{xx}+b_1u_t+b_2u_x+cu-f). \tag{3}$$

Where the differential operator L is

$L=\frac{\partial^2}{\partial t^2}$

with the inverse

$$L^{-1}(.)=\int_0^t\int_0^t(.)dtdt. \tag{4}$$

operating with L^{-1} on the both sides of equation (4) and using the initial conditions yields to

$$u(x,t)=f_0(x)+tf_1(x)+L^{-1}(\frac{1}{a_{11}}(2a_{12}u_{xx}+a_{22}u_{tx}+b_1u_t+b_2u_x+cu-f)) \tag{5}$$

To solve Equation.(3) by Adomian decomposition method ,well addressed in [2,3],let's consider the solution as the summation of a series,say; $u=\sum_{n=0}^{\infty}u_n(x,t)$

So that the components u_m will be determined recursive.And the integrand on the right side as the sum a series as

$$\frac{1}{a_{11}}(2a_{12}u_{xx}+a_{22}u_{tx}+b_1u_t+b_2u_x+cu-f)=\sum_{n=0}^{\infty}A_n(u_0,u_1,u_2,\ldots,u_n) \tag{6}$$

Where $A_n(u_0,u_1,u_2,\ldots,u_n)$ are called Adomian polynomials and should be computed.By using an Alternate Algorithm for computing Adomian polynomial [4] we have

$$\frac{1}{a_{11}}(2a_{12}u_{nxx}+a_{22}u_{ntx}+b_1u_{nt}+b_2u_{nx}+cu_n-f)=A_n(u_0,u_1,u_2,\ldots,u_n). \tag{7}$$

Substituting $u = \sum_{n=0}^{\infty} u_n$ and $A_n(u_0, u_1, u_2, \dots, u_n)$ in (5), we drive

$$\sum_{n=0}^{\infty} u_n = f_0(x) + tf_1(x) + \frac{1}{a_{11}} L^{-1} f \tag{8}$$

$$+ \sum_{n=0}^{\infty} L^{-1}(\frac{1}{a_{11}}(2a_{12}u_{nxx} + a_{22}u_{ntx} + b_1 u_{nt} + b_2 u_{nx} + cu_n)$$

Therefor from (8) the following procedure can be defined

$$u_0(x,t) = f_0(x) + tf_1(x),$$

$$u_{n+1}(x,t) = \int_0^t \int_0^t A_n(u_0, u_1, u_2, \dots, u_n) dtdt, \quad n = 0, 1, 2, \dots \tag{9}$$

In some cases, having some terms we can recognize the general term and the solution can be denoted as a series. If the series can be recognized as the series of a specified function, the solution can be introduced as an analytic function.In other cases we can have the solution as summation of a few terms of below series;

$$u = \sum_{k=0}^{n} u_k$$

3 Numerical results

To illustrate the method five examples in different spaces are presented. Consider the following partial differential equations, which are known as Cauchy problems [1]

Example 1. Cauchy problem for infinite string

$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \quad -\infty < x, y < \infty \quad t > 0$

$u(x,0) = cos(bx + cy)$

$u_t(x,0) = sin(bx + cy)$

By using (9)

$u_0(x,t) = cos(bx + cy) + tsin(bx + cy)$

$u_{n+1}(x,t) = a^2 \int_0^t \int_0^t (u_{0xx} + u_{0yy} + u_{0zz}) dtdt \qquad n = 0, 1, 2, 3, \ldots$

we have

$u_1 = -a^2[\frac{t^2}{2!}(b^2 + c^2)cos(bx + cy) + \frac{t^3}{3!}(b^2 + c^2)sin(bx + cy)]$

$u_2 = a^4[\frac{t^4}{4!}(b^2 + c^2)^2cos(bx + cy) + \frac{t^5}{5!}(b^2 + c^2)^2sin(bx + cy)].$

The general term can be recognized as

$u_n = (-1)^n a^{2n}[(b^2 + c^2)^{2n}cos(bx + cy) + \frac{t^{2n+1}}{(2n+1)!}(b^2 + c^2)^{2n+1}sin(bx + cy).$

Hence the solution is

$u = \sum_{k=0}^{\infty} u_k = cos(bx+cy)(1 - \frac{a^2(b^2+c^2)t^2}{2!} + \frac{a^4(b^2+c^2)^2t^4}{4!} - \ldots) + \frac{1}{a\sqrt{a^2+b^2}} sin(bx + cy)$

$(a\sqrt{b^2 + c^2}t - \frac{a^3(b^2+c^2)}{3!}t^3 - \frac{a^5(b^2+c^2)^2\sqrt{b^2+c^2}}{5!}t^5 + \ldots)$

and therefore the exact solution is

$u(x,t) = cos(bx + cy)cos(a\sqrt{b^2 + c^2}t) + \frac{1}{a\sqrt{b^2+c^2}} sin(bx + cy)(sin(a\sqrt{b^2 + c^2}t)$

Example 2. In this example we solve a partial differential equation in two dimentions

$u_{tt} = u_{xx} + e^{-t} \qquad -\infty < x < +\infty$

$u(x,0) = sinx$

$u_t(x,0) = cosx \qquad -\infty < x < +\infty$

By using (9) the Adomian scheme would be as follows

$u_0 = sinx + tcosx$

$u_{n+1} = \int_0^t \int_0^t (u_{nxx} + e^{-t}) dtdt \qquad n = 0, 1, 2, 3, \ldots$

For the first few values of n, we have

$u_1 = \int_0^t \int_0^t -(sinx + tcosx) + e^{-t}dtdt = -(sinx\frac{t^2}{2!} + cosx\frac{t^3}{3!}) + e^{-t} + t - 1$

$u_2 = \int_0^t \int_0^t (sinx\frac{t^2}{2!} + cosx\frac{t^3}{3!} + e^{-t} + t - 1)dtdt = sinx\frac{t^4}{4!} + cosx\frac{t^5}{5!} + e^{-t} - 1 + t - \frac{t^2}{2!} + \frac{t^3}{3!}$

$\vdots$

$u_{n+1} = sinx\frac{t^{2n}}{2n!} + cosx\frac{t^{2n+1}}{(2n+1)!} + e^{-t} - 1 + t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \ldots - \frac{t^{2n}}{(2n)!}) + \frac{t^{2n+1}}{(2n+1)!}$

$u = \sum_{n=0}^{\infty} u_n = sinx(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots - \frac{t^{2n}}{2n!}) + cosx(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots + \frac{t^{2n+1}}{(2n+1)!}) + e^{-t} + t - 1 + n(e^{-t}) - n(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots + \frac{t^n}{n!})$

$\lim_{n\to\infty} u_n = sinx\ cost + cosx\ sint + e^{-t} + t - 1$

$u = sin(x + t) + e^{-t} + t - 1$

Example 3. Let's consider the following P.D.E

$u_{tt} = u_{xx} + u_{yy} + tsiny \qquad -\infty < x, y < \infty \quad t > 0$

$u(x, y, 0) = x^2$

$u(x, y, 0) = siny.$

By using (9) the Adomian scheme would be as follows

$u_0 = x^2 + tsiny$

$u_{n+1} = L^{-1}(u_{nxx} + u_{nyy} + tsiny)$

For the first few values of n, we have

$u_1 = \int_0^t \int_0^t 2 - tsiny + tsiny = t^2$

$u_2 = \int_0^t \int_0^t 0 = 0$

$\vdots$

$u_{n+1} = 0$

So the exact solution is

$u = x^2 + t^2 + tsiny$

Example 4.A partial differential equation, with initial conditions, in three dimensions is discussed here

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} + 2xyz \qquad -\infty < x, y, z < \infty, t > 0$$

$$u(x, y, z, 0) = x^2 + y^2 - 2z^2,$$

$$u_t(x, y, z, 0) = 0. \qquad -\infty < x, y, z < \infty$$

By using (9)we obtain

$$u_0 = x^2 + y^2 - 2z^2 + t$$

$$u_{n+1} = L^{-1}(u_{nxx} + u_{nyy} + u_{nzz} + 2xyz)$$

For the first few values of n, we have

$$u_1 = \int_0^t \int_0^t 2 + 2 - 4 + tdtdt = \frac{t^3}{3!}$$

$$u_2 = \int_0^t \int_0^t 0dtdt$$

$$u_3 = 0$$

$$\vdots$$

$$u_n = 0.$$

And therefore the exact solution is

$$u(x, t) = x^2 + y^2 - 2z^2 + \frac{t^3}{3!}$$

Example 5. The following P.D.E, with initial conditions is considered

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} + te^{5x} sin(3y) cos(4z)$$

$$u(x, 0) = e^{6x+8y} cos(10z)$$

$$u_t(x, 0) = e^{3y+4z} sin(5x).$$

By using (9) we get

$$u_0 = e^{6x+8y} cos(10z) + te^{3y+4z} sin(5x)$$

$u_{n+1} = L^{-1}(u_{nxx} + u_{nyy} + u_{nzz} + te^{5x}sin(3y)cos(4z)). \quad n = 0, 1, 2, 3, \ldots$

For the first few values of n,we have

$u_1 = \frac{t^2}{2!}e^{5x}sin(3y)cos(4z)$

$u_2 = 0$

$u_3 = 0$

$\vdots$

and therefore the exact solution would be as

$u(x,t) = e^{6x+8y}cos(10z) + te^{3y+4z}sin(5x) + \frac{t^3}{3!}e^{5x}sin(3y)cos(4z).$

4 Conclusions

The main goal of this article has been to derive an analytical solution for hyperbolic equations. We have achieved this goal by applying Adomian decomposition method.The results show that this method is very simple and effective for such equations.

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