

Numerical Approach to Fixed Point Theorems

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Abstract. Almost since L.E. Brouwer (1912) proved a remarkable result saying that any continuous function from the n -dimensional unit ball to itself has a fixed point, a point that is mapped by the function into itself. The Brouwer fixed point theorem was one of the early major achievements of algebraic topology. This celebrated theorem has been generalized in several ways. Nowadays, the Brouwer, Kakutani, and Tarski theorems have become the most often used tools in economics, game theory and numerical analysis. In this paper, we give an elementary fixed point theorems and an algorithm to resolve the problem of fixed point theorems.

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1. INTRODUCTION

Almost since L.E. Brouwer ([3], 1912) proved a remarkable result saying that any continuous function from the n -dimensional unit ball to itself has a fixed point, a point that is mapped by the function into itself. The Brouwer fixed point theorem was one of the early major achievements of algebraic topology. This celebrated theorem has been generalized in several ways, for général surveys of the literature see, e.g., R. Smart ([16], 1974) and V.I. Istratescu ([11], 1981). For instance, J. Schauder ([14], 1930) generalized the theorem to Banach spaces, S. Kakutani ([12], 1941) gave an extension to upper semi-continuous point-to-set mappings.

Existence results of fixed points in case the function is not continuous were given by A. Tarski ([18], 1955) and J. Caristi ([4], 1976). Tariski's theorem is restricted to functions on a sulattice satisfying some monotonicity condition. Caristi's theorem concerns functions satisfying a non-expansion condition. However, it should be

noticed that both these theorems do not cover Brouwer's theorem, because a continuous function does not need to satisfy the conditions in Tarski's or Caristi's theorem. Another major development during the last few decades is about the computation of fixed points of a continuous function or upper semi-continuous point-to-set mapping, see, e.g., H.E. Scarf ([13], 1973), M.J. Todd ([19], 1976), E.L. Allgower and K. Georg([1], 1990), and Z. Yang ([20], 1999; [21], 2004).

Nowadays, the Brouwer, Kakutani, and Tarski theorems have become the most often used tools in economics, game theory and numerical analysis, see, e.g., K.J. Arrow and F.H. Hahn ([2], 1971), D. Fudenberg and J. Tirole ([8], 1991), P.J.J. Herings ([10], 1996) and N. Daili ([5], 2000). Also on the practical frontier fixed point methods are used by applied economists to analyse equilibrium models, for instance, to study the effects of policy and technical changes, see e.g., J.B. Shoven and J. Whalley ([15], 1992).

In this paper, we give an elementary fixed point theorems and an algorithm to resolve the problem of fixed point theorems.

2. Functional Aspects

Definition 1. *Let E be a linear normed space. Let $F : E \rightarrow E$. A map F is called contracting if there exists a constant $\alpha < 1$ such that*

$$(2.1) \quad \|F(\zeta) - F(\xi)\| \leq \alpha \|\zeta - \xi\| \quad \forall \zeta, \xi \in E$$

Theorem 1. *A contracting map F in Banach space B has a fixed point; that is, there is $\zeta \in B$ such that*

$$F(\zeta) = \zeta.$$

Proof. Let $\zeta_0 \in B$ and define a sequence $\{\zeta_n\} \subset B$ by

$$\zeta_n = F^n(\zeta_0), \quad n = 1, 2, \dots$$

Then if $n \geq m$, we have

$$\begin{aligned} \|\zeta_n - \zeta_m\| &\leq \sum_{j=m+1}^n \|\zeta_j - \zeta_{j-1}\| = \sum_{j=m+1}^n \|F^{j-1}(\zeta_0) - F^{j-1}(\zeta_0)\| \\ &\leq \sum_{j=m+1}^n \alpha^{j-1} \|\zeta_1 - \zeta_0\| \leq \alpha^m \frac{\|\zeta_1 - \zeta_0\|}{1-\alpha}. \end{aligned}$$

The last term tends to zero as m tends to infinity. Thus $\{\zeta_n\}_n$ is a Cauchy sequence and, since B is complete space, then converges to one element $\zeta \in B$. That is so F is also continuous and we have

$$F(\zeta) = \lim_{(n \rightarrow +\infty)} F(\zeta_n) = \lim_{(n \rightarrow +\infty)} \zeta_{n+1} = \zeta,$$

then ζ is a fixed point of F . Uniqueness of ζ results immediately from (2.1). ■

Theorem 2. (Schauder)([5], [6]) *Let C be a non-empty convex and compact set of the linear normed space E , and F is a continuous map from $C \rightarrow C$. Then F has at least a fixed point.*

Proof. See, e.g, N. Dunford and J. T. Schwartz ([6], 1958, Theorem 5, p. 456), R. Edwards ([7], 1965, Theorem 3.6.1, p. 161). ■

The proof is based on the following Brouwer fixed point theorem (similar to Schauder fixed point theorem in finite dimension):

Theorem 3. (Brouwer Fixed Point Theorem) *Let C be a non-empty convex set and compact in a finite dimensional space. Let F be a continuous map from C into itself. Then F has at least a fixed point.*

3. Numerical Aspects

Let $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a map.

Remark 1. *All norms in \mathbb{R}^N are equivalent and thus notions of open and closed sets do not depend of the chosen norm in \mathbb{R}^N .*

Remark 2. *A subset E of \mathbb{R}^N can be neither open nor closed. We can take $N = 1$ and see case where E is successively one of intervals : $[0, 1]$, $]0, 1[$ and $[0, 1[$.*

Proposition 4. *The following two properties are equivalent:*

- a_1) $E \subset \mathbb{R}^N$ is closed;
- a_2) for all sequence $\{\zeta_n\}_{n \geq 1} \subset E$ which converges to $\zeta \in \mathbb{R}^N$, then $\zeta \in E$.

Definition 2. *We call N -vector ζ a fixed point of F if $\zeta = F(\zeta)$.*

Definition 3. *Let $E \subset \mathbb{R}^N$. We call F a contracting map in E (for the norm $|\cdot|$), if there exists a constant $\alpha < 1$ such that*

$$|F(\zeta) - F(\xi)| \leq \alpha |\zeta - \xi|$$

for all N -vectors ζ and ξ of E .

Remark 3. • *If F is contracting in E , then F is continuous in E ;*
 • *F can be contracting in E for a norm $|\cdot|$ and not contracting for an other norm.*

Theorem 5. *Let $E \subset \mathbb{R}^N$ be a non-empty closed subset. Suppose*

- $(H_1) : F(E) \subset E$,
- $(H_2) : F$ is contracting in E .

Then F has unique fixed point ζ in E . Furthermore, if $\zeta_0 \in E$ and $\zeta_{k+1} = F(\zeta_k)$, for $k = 0, 1, 2, \dots$, then

$$\lim_{(k \rightarrow +\infty)} |\zeta - \zeta_k| = 0.$$

Proof. Existence: suppose (H_1) and (H_2) satisfied and let any $\zeta_0 \in E$. Put

$$\zeta_{k+1} = F(\zeta_k), \quad k = 0, 1, 2, \dots$$

We verify that $\zeta_k \in E$, $\forall k = 0, 1, 2, \dots$ (hypothesis (H_1)) and we have for $k \geq 1$:

$$|\zeta_{k+1} - \zeta_k| = |F(\zeta_k) - F(\zeta_{k-1})| \leq \alpha |\zeta_k - \zeta_{k-1}|$$

where $\alpha < 1$ is the contracting constant of F in E . Recursively, we obtain the relation:

$$|\zeta_{k+1} - \zeta_k| \leq \alpha^k |\zeta_1 - \zeta_0|, \quad \forall k \geq 0.$$

Now, let $k \geq 0$, $s \geq 1$ and estimate

$$\begin{aligned} |\zeta_{k+s} - \zeta_k| &\leq |\zeta_k - \zeta_{k+1}| + |\zeta_{k+1} - \zeta_{k+2}| + \dots + |\zeta_{k+s-1} - \zeta_{k+s}| \\ &\leq (\alpha^k + \alpha^{k+1} + \dots + \alpha^{k+s-1}) |\zeta_1 - \zeta_0| = \alpha^k \left(\sum_{i=0}^{s-1} \alpha^i \right) |\zeta_1 - \zeta_0| \\ &= \alpha^k \frac{1-\alpha^s}{1-\alpha} |\zeta_1 - \zeta_0|. \end{aligned}$$

Since $\alpha < 1$, $\{\zeta_k\}_{k \geq 1}$ is a Cauchy sequence, namely

$$\lim_{(k \rightarrow +\infty, m \rightarrow +\infty)} |\zeta_k - \zeta_m| = 0;$$

thus it converges. Let $\zeta \in \mathbb{R}^N$ such that

$$\lim_{(k \rightarrow +\infty)} |\zeta - \zeta_k| = 0.$$

Since E is closed (hypothesis), we will have $\zeta \in E$ (see Proposition 3.1). Thus

$$\begin{aligned} |\zeta - F(\zeta)| &\leq |\zeta - \zeta_k| + |\zeta_k - F(\zeta)| = |\zeta - \zeta_k| + |F(\zeta_{k-1}) - F(\zeta)| \\ &\leq |\zeta - \zeta_k| + \alpha |\zeta - \zeta_{k-1}|; \end{aligned}$$

by taking the limit when k tends to infinity, we obtain $\zeta = F(\zeta)$, which proves that ζ is a fixed point of F in E .

Uniqueness: suppose $\xi \in E$ is an other fixed point of F . We have

$$|\xi - \zeta| = |F(\xi) - F(\zeta)| \leq \alpha |\xi - \zeta|$$

and since $\alpha < 1$, we obtain $\zeta = \xi$, which proves the uniqueness fixed point and theorem holds. ■

Remark 4. *Relations*

$$\zeta_{k+1} = F(\zeta_k), \quad \text{for } k = 0, 1, 2, \dots$$

in Theorem 3.2 give an algorithm to find and calculate the fixed point ζ of F in E .
If

$$\varepsilon_k = |\zeta - \zeta_k|$$

is the error in k^{th} step, we will have

$$\varepsilon_{k+1} = |\zeta - \zeta_{k+1}| = |F(\zeta) - F(\zeta_k)| \leq \alpha |\zeta - \zeta_k| = \alpha \varepsilon_k.$$

Thus, to each step, we reduce error with factor $\alpha < 1$; in this case we have linear convergence.

Definition 4. If $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ has continuous components:

$$F_1(\zeta) = F_1(\zeta_1, \zeta_2, \dots, \zeta_N), F_2(\zeta) = F_2(\zeta_1, \zeta_2, \dots, \zeta_N), \dots, F_N(\zeta) = F_N(\zeta_1, \zeta_2, \dots, \zeta_N)$$

with first partial derivatives continuous, we define the $N \times N$ -matrix $DF(\zeta)$ at point ζ by

$$DF(\zeta) = \begin{pmatrix} \frac{\partial F_1}{\partial \zeta_1}(\zeta) & \frac{\partial F_1}{\partial \zeta_2}(\zeta) & \cdot & \cdot & \cdot & \frac{\partial F_1}{\partial \zeta_N}(\zeta) \\ \frac{\partial F_2}{\partial \zeta_1}(\zeta) & \frac{\partial F_2}{\partial \zeta_2}(\zeta) & \cdot & \cdot & \cdot & \frac{\partial F_2}{\partial \zeta_N}(\zeta) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial F_N}{\partial \zeta_1}(\zeta) & \frac{\partial F_N}{\partial \zeta_2}(\zeta) & \cdot & \cdot & \cdot & \frac{\partial F_N}{\partial \zeta_N}(\zeta) \end{pmatrix};$$

$DF(\zeta)$ is called jacobian matrix of F at point ζ .

Definition 5. a) If B is an $N \times N$ -matrix of eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ (real or complex), then the spectral radius $\rho(B)$ of B is defined by

$$\rho(B) = \max_{i=1}^N |\lambda_i|,$$

where $|\lambda_i|$ is the norm of λ_i .

b) The norm $|B|$ of B subordinate to the norm $|\cdot|$ of \mathbb{R}^N is defined by

$$|B| = \max_{\zeta \in \mathbb{R}^N, \zeta \neq 0} \frac{|B\zeta|}{|\zeta|}.$$

Corollary 6. We have

$$\rho(B) \leq |B|$$

for all norm $|\cdot|$ of \mathbb{R}^N .

Remark 5. *The spectral radius is not a norm and we can have $\rho(B) = 0$ without B be zero. We can take for example $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.*

We have the following lemma:

Lemma 7. *Let B be an $N \times N$ -matrix and let $\varepsilon > 0$. There exists a norm $|\cdot|$ (which depends of B and ε) such that*

$$|B| \leq \rho(B) + \varepsilon.$$

Theorem 8. *Let $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous mapping with first partial derivatives continuous and let ζ be a fixed point of F . If $\rho(DF(\zeta)) < 1$, then there exists $r > 0$ such that if $\zeta_0 \in \mathbb{R}^N$, $\zeta_0 \in B[\zeta; r]$ (= the closed ball of center ζ and radius r for the norm $|\cdot|$), the sequence $\{\zeta_k\}_{k \geq 1}$ defined by*

$$\zeta_{k+1} = F(\zeta_k), \quad k = 0, 1, 2, \dots$$

is enclosed in $B[\zeta; r]$ and converges to ζ .

Proof. By the previous lemma and continuity of $DF(\eta)$, we can find a norm $|\cdot|$ of \mathbb{R}^N and a number $r > 0$ such that

$$|DF(\eta)| \leq \frac{1 + \rho(DF(\zeta))}{2} \equiv \alpha < 1, \quad \forall \eta \in B[\zeta; r].$$

If ξ and $\eta \in B[\zeta; r]$, we verify that

$$F(\xi) - F(\eta) = \int_0^1 DF(\eta + t(\xi - \eta))(\xi - \eta) dt$$

and thus

$$\begin{aligned} |F(\xi) - F(\eta)| &= \left| \int_0^1 DF(\eta + t(\xi - \eta))(\xi - \eta) dt \right| \\ &\leq \int_0^1 |DF(\eta + t(\xi - \eta))| |\xi - \eta| dt \leq \alpha |\xi - \eta|. \end{aligned}$$

Hence, F is contracting in $B[\zeta; r]$. Furthermore, if $\xi \in B[\zeta; r]$, then

$$|F(\xi) - \zeta| = |F(\xi) - F(\zeta)| \leq \alpha |\xi - \zeta| \leq r,$$

which proves that $F(B[\zeta; r]) \subset B[\zeta; r]$. Theorem 3.2 enable to conclude. ■

4. Fixed Point Algorithm

Consider the following equation:

$$(4.1) \quad F(\zeta) = \zeta$$

We find the fixed points of F by the following algorithm:

Algorithm:

Step 0: Given a starting value ζ_0 and choose $\varepsilon > 0$.

Step 1: Calculate $\zeta_{k+1} = F(\zeta_k)$, for $k \geq 0$.

Step 2: If $|\zeta_{k+1} - F(\zeta_{k+1})| \leq \varepsilon$, stop, since ζ_{k+1} is an inquired approach solution of equation (4.1); else, we go on.

This algorithm converges under some conditions given by the following theorem:

Theorem 9. *Suppose the mapping F continuously differentiable. Then, if the differential of F (denoted F') is bounded by α , with $0 \leq \alpha < 1$, i.e. if*

$$(4.2) \quad \forall \zeta \in \mathbb{R}^N, \quad |F'(\zeta)| \leq \alpha < 1$$

algorithm converges to a solution of (4.1).

Proof. By (4.2) we have

$$|\zeta_{n+1} - \zeta_n| = |F(\zeta_n) - F(\zeta_{n-1})| \leq \alpha |\zeta_n - \zeta_{n-1}|,$$

which gives recursively

$$|\zeta_{n+1} - \zeta_n| \leq \alpha^n |\zeta_1 - \zeta_0|.$$

And by sommation, we obtain

$$|\zeta_{k+p} - \zeta_k| \leq \sum_{n=k}^{k+p-1} |\zeta_{n+1} - \zeta_n| \leq \left(\sum_{n=k}^{k+p-1} \alpha^n \right) |\zeta_1 - \zeta_0| \leq \frac{\alpha^k}{1 - \alpha} |\zeta_1 - \zeta_0|,$$

inequality proves that $\{\zeta_k\}_k$ is a Cauchy sequence in \mathbb{R}^N ; thus this last is convergente of limit denoted ζ^* . By definition and continuity of F , we deduce that

$$0 = \zeta_{k+1} - F(\zeta_k) \rightarrow 0 = \zeta^* - F(\zeta^*),$$

which proves that ζ^* is a solution of (4.1). ■

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