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Subclasses of Analytic Functions Involving a Certain Family of Linear Operators

C. Selvaraj

Department of Mathematics, Presidency College (Autonomous)
Chennai-600 005, Tamilnadu, India
pamc9439@yahoo.co.in

K. R. Karthikeyan

Department of Mathematics, R. M. K. Engineering College
R. S. M. Nagar, Kavaraipettai-601206, Tamilnadu, India
kr_karthikeyan1979@yahoo.com

Abstract

This paper introduces a new class of analytic functions with negative coefficients defined using a family of linear operators. Necessary and sufficient conditions, coefficient estimates, distortion bounds and radius of convexity are determined. Also integral means inequalities are obtained for fractional derivatives of order $m + \delta$.

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1 Introduction, Definitions And Preliminaries

Let $\mathcal{A}$ be the class of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the open unit disc $\mathcal{U} = \{z : | z | < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}$. The Hadamard product of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathcal{A}$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n\, b_n\, z^n. \tag{1}$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathcal{Z}_0^-$; $\mathbb{Z}_0^- = 0, -1, -2, \ldots$; $j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$$ {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ \ldots (\alpha_q)_n}{(\beta_1)_n \ \ldots (\beta_s)_n} \frac{z^n}{n!} $$

$$ (q \leq s+1;\ q, s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\};\ z \in \mathcal{U}), $$

where $\mathcal{N}$ denotes the set of positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2)\ \ldots (x+k-1) & if\, k \in N = \{1, 2, , \ldots\}. \end{cases} $$

Corresponding to a function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$ h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{2} $$

Dziok and Srivastava considered a linear operator

$$ H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \quad : \quad \mathcal{A} \longrightarrow \mathcal{A} $$

defined by Hadamard product (or convolution):

$$ H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) f(z) := h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). $$

The linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) f(z)$ includes (as its special cases) various other linear operators which were introduced and studied by Hohlov [9], Carlson and Shaffer [3], Ruscheweyh [15] and so on.

Corresponding to the function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by (2), we define a function $h_{\mu,p}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ given by

$$ z + \sum_{n=2}^{\infty} \frac{(n+\mu)^p}{(\mu+1)^p} z^n * h_{\mu,p}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \tag{3} $$

$$ = h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z). $$

Analogous to $H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$, Selvaraj et. al. considered a linear operator $\mathcal{G}^p_\mu(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ on $\mathcal{A}$ as follows:

$$\mathcal{G}^p_\mu(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)f(z) = h_{\mu,p}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z) \tag{4}$$

$(\alpha_i, \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; i = 1, \dots, q; j = 1, \dots, s; \mu \neq -1; p \in \mathcal{N}_0 = \{0, 1, 3, \dots\})$.

For convenience, we write

$$\mathcal{G}^p_{\mu,q,s}(\alpha_1) = \mathcal{G}^p_\mu(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z). \tag{5}$$

It can be easily verified from the definition of (4) that

$$z(\mathcal{G}^{p+1}_{\mu,q,s}(\alpha_1)f(z))' = (\mu+1)\mathcal{G}^p_{\mu,q,s}(\alpha_1)f(z) - \mu\,\mathcal{G}^{p+1}_{\mu,q,s}(\alpha_1)f(z) \tag{6}$$

and

$$z(\mathcal{G}^p_{\mu,q,s}(\alpha_1)f(z))' = \alpha_1\mathcal{G}^p_{\mu,q,s}(\alpha_1+1)f(z) - (\alpha_1-1)\mathcal{G}^p_{\mu,q,s}(\alpha_1)f(z). \tag{7}$$

To provide a unified approach to the study of various properties of the certain subclasses of $\mathcal{A}$, we now introduce the most generalized subclass of $\mathcal{A}$ by using the operator (4).

Definition 1.1 *For any non-zero complex number λ, $0 \le \gamma < 1$ and $k \ge 0$, a function $f \in \mathcal{A}$ is said to be in $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ if and only if it satisfies the condition*

$$\mathfrak{Re}\left\{\frac{\lambda\mathcal{G}^p_{\mu,q,s}(\alpha_1+1)f(z)}{\mathcal{G}^p_{\mu,q,s}(\alpha_1)f(z)} - (\lambda-1)\right\} > k\left|\frac{\lambda\mathcal{G}^p_{\mu,q,s}(\alpha_1+1)f(z)}{\mathcal{G}^p_{\mu,q,s}(\alpha_1)f(z)} - \lambda\right| + \gamma. \tag{8}$$

The family $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\lambda = \alpha_1$ and for appropriate choices of the parameters $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ reduces to $\mathcal{L}(a, c; \alpha, \beta)$[5]. We note that the family $\mathcal{S}^*(\alpha)$ of starlike function of order $\alpha\,(0 \le \alpha < 1)$[4, 6], the family $\mathcal{C}(\alpha)$ of convex function of order $\alpha\,(0 \le \alpha < 1)$[4, 6], $k-UCV(\alpha)$[2], $k-UST(\alpha)$ and many other well known subclasses of $\mathcal{S}$ (see also the work of Kanas and Srivastava [10], Goodman [7, 8] and Rønning [13, 14]) are the special cases of $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$.

Further we define $\mathcal{I}_T(\alpha_1, q, s : k, \gamma, \lambda) = \mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda) \cap T$, where T is the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \ge 0, \quad \forall\, n \ge 2 \tag{9}$$

was introduced and studied by Silverman [16].

In this paper we provide necessary and sufficient conditions, coefficient bounds, extreme points, radius of convexity, starlikeness and close-to-convexity, closure theorem for functions in $\mathcal{I}_T(\alpha_1, q, s : k, \gamma, \lambda)$.

2 Characterization

We employ the technique adapted by Aqlan et al. [1] to find the coefficient estimates for our class.

Theorem 2.1 *$f(z) \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$ if and only if*

$$\sum_{n=2}^{\infty} \Big(\lambda(1+k)(\alpha_1+1)_{n-1} + \big[(1-\gamma) - \lambda(1+k)\big](\alpha_1)_{n-1} \Big) \Psi(n)\, a_n \leq (1-\gamma) \tag{10}$$

where $\Psi(n) = \dfrac{(\mu+1)^p (\alpha_2)_{n-1} \ldots (\alpha_q)_{n-1}}{(n+\mu)^p (\beta_1)_{n-1} \ \ldots \ (\beta_s)_{n-1}}$. This result is sharp.

Proof. By definition $f(z) \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$ if and only if the condition (8) is satisfied. Upon the fact that

$$\mathfrak{Re}\{\lambda w - (\lambda - 1)\} = k \mid \lambda(w-1) \mid + \gamma$$

$$\Longleftrightarrow \mathfrak{Re}\big\{(\lambda w - (\lambda-1))(1+ke^{i\theta}) - ke^{i\theta}\big\} > \gamma \quad -\pi \leq \theta < \pi.$$

Equation (8) can be written as

$$\mathfrak{Re}\bigg\{ \bigg(\frac{\lambda \mathcal{G}^p_{\mu,q,s}(\alpha_1+1) f(z)}{\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)} - (\lambda-1) \bigg)(1+ke^{i\theta}) - ke^{i\theta} \bigg\} > \gamma$$

or equivalently

$$\mathfrak{Re}\bigg\{ \frac{\big(\lambda \mathcal{G}^p_{\mu,q,s}(\alpha_1+1) f(z) - (\lambda-1)\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)\big)(1+ke^{i\theta})}{\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)}$$

$$\frac{-ke^{i\theta}\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)}{\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)} \bigg\} > \gamma. \tag{11}$$

Now we let $A(z) =$

$$\big(\lambda \mathcal{G}^p_{\mu,q,s}(\alpha_1+1) f(z) - (\lambda-1)\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)\big)(1+ke^{i\theta}) - ke^{i\theta}\mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)$$

and let $B(z) = \mathcal{G}^p_{\mu,q,s}(\alpha_1) f(z)$.
Then (11) is equivalent to $\mid A(z) + (1-\gamma)B(z) \mid > \mid A(z) - (1+\gamma)B(z) \mid$ for $0 \leq \gamma < 1$. For $A(z)$ and $B(z)$ as above, we have $\mid A(z) + (1-\gamma)B(z) \mid$

$$= \bigg| (2-\gamma)z - \sum_{n=2}^{\infty} \Big(\lambda(\alpha_1+1)_{n-1} + \big[(2-\gamma) - \lambda\big](\alpha_1)_{n-1} \Big) \Psi(n)\, a_n z^n$$

$$-ke^{i\theta}\sum_{n=2}^{\infty}\left(\lambda\big[(\alpha_1+1)_{n-1}-(\alpha_1)_{n-1}\big]\right)\Psi(n)\,a_nz^n\Bigg|$$

$$\geq (2-\gamma)\mid z\mid$$

$$-\sum_{n=2}^{\infty}\left(\lambda(1+k)(\alpha_1+1)_{n-1}+\big[(2-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\right)\Psi(n)\,a_n\mid z\mid^n,$$

and similarly
$\mid A(z)-(1+\gamma)B(z)\mid$

$$<\gamma\mid z\mid+\sum_{n=2}^{\infty}\left(\lambda(1+k)(\alpha_1+1)_{n-1}+\big[-\gamma-\lambda(1+k)\big](\alpha_1)_{n-1}\right)\Psi(n)\,a_n\mid z\mid^n .$$

Therefore, $\mid A(z)+(1-\gamma)B(z)\mid-\mid A(z)-(1+\gamma)B(z)\mid$

$\geq 2(1-\gamma)\mid z\mid$

$$-2\sum_{n=2}^{\infty}\left(\lambda(1+k)(\alpha_1+1)_{n-1}+\big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\right)\Psi(n)\,a_n\mid z\mid^n\geq 0,$$

or

$$\sum_{n=2}^{\infty}\left(\lambda(1+k)(\alpha_1+1)_{n-1}+\big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\right)\Psi(n)\,a_n\leq(1-\gamma)$$

which yields (10).
On the other hand, we must have

$$\mathfrak{Re}\left\{\left(\frac{\lambda\mathcal{G}^p_{\mu,\,q,\,s}(\alpha_1+1)f(z)}{\mathcal{G}^p_{\mu,\,q,\,s}(\alpha_1)f(z)}-(\lambda-1)\right)(1+ke^{i\theta})-ke^{i\theta}\right\}>\gamma\quad -\pi\leq\theta<\pi.$$

Upon choosing the values of z on the positive real axis where $0\leq\mid z\mid=r<1$, the above inequality reduces to

$$\mathfrak{Re}\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}\left(\lambda(\alpha_1+1)_{n-1}+\big[(1-\gamma)-\lambda\big](\alpha_1)_{n-1}\right)\Psi(n)\,a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(\alpha)_1\Psi(n)\,a_nr^{n-1}}\right.$$

$$\left.\frac{-ke^{i\theta}\sum_{n=2}^{\infty}\left(\lambda\big[(\alpha_1+1)_{n-1}-(\alpha_1)_{n-1}\big]\right)\Psi(n)\,a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(\alpha)_1\Psi(n)\,a_nr^{n-1}}\right\}\geq 0.$$

Since $\mathfrak{Re}(-e^{i\theta}) \geq - \mid e^{i\theta} \mid = -1$, the above inequality reduces to

$$\mathfrak{Re}\left\{ \frac{(1-\gamma)-\sum_{n=2}^{\infty}\Big(\lambda(1+k)(\alpha_1+1)_{n-1}+\big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\Big)\Psi(n)\, a_n r^{n-1}}{1-\sum_{n=2}^{\infty}(\alpha)_1\Psi(n)\, a_n r^{n-1}} \right\} \geq 0.$$

Letting, $r \to 1^-$, we get the desired result.
Finally, the function $f(z)$ given by

$$f(z) = z - \frac{1-\gamma}{\Big(\lambda(1+k)(\alpha_1+1)_{n-1} + \big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\Big)\Psi(n)} z^n \tag{12}$$

for $(n \geq 2)$ is an extremal function for the assertion of Theorem (2.1).

Corollary 2.2 *If $f(z) \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$, then*

$$a_n \leq \frac{1-\gamma}{\Bigg(\lambda(1+k)(\alpha_1+1)_{n-1} + \big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\Bigg)\Psi(n)}. \tag{13}$$

Equality being attained for the function $f(z)$ given by (12).

Remark 2.3 *For different choices of p, q, s, k, μ, λ, γ, α_1, α_2 and β_1 as stated in Theorem (2.1) leads to necessary and sufficient condition for a function f of the form (9) to be in the classes $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $k-UCV(\alpha)$, $k-UST(\alpha)$ and $\mathcal{L}_T(a, c;\, \alpha, \beta)$.*

For convenience we shall henceforth denote

$$\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda) = \Big(\lambda(1+k)(\alpha_1+1)_{n-1} + \big[(1-\gamma)-\lambda(1+k)\big](\alpha_1)_{n-1}\Big)\Psi(n). \tag{14}$$

3 Growth and Distortion Theorems

Theorem 3.1 *Let $f \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$. If $\big\{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)\big\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $\mid z \mid = r < 1$*

$$r - \frac{1-\gamma}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r^2 \leq \mid f(z) \mid \leq r + \frac{1-\gamma}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r^2 \tag{15}$$

and if $\big\{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)/n\big\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $\mid z \mid = r < 1$

$$1 - \frac{2(1-\gamma)}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r \leq \mid f^{'}(z) \mid \leq 1 + \frac{2(1-\gamma)}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r. \tag{16}$$

The results (15) and (16) are sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\gamma}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} z^2. \tag{17}$$

Proof. We only prove the right hand side inequality in (15), since the other inequality can be justified using similar arguments. Since $f \in \mathcal{I}_T(\alpha_1, q, s : k, \gamma, \lambda)$, by Theorem (2.1),

$$\sum_{n=2}^{\infty} \left(\lambda(1+k)(\alpha_1+1)_{n-1} + \big[(1-\gamma) - \lambda(1+k)\big](\alpha_1)_{n-1} \right) \Psi(n)\, a_n \le (1-\gamma).$$

Now

$$\begin{aligned} \xi_2(\alpha_1, q, s:\ k, \gamma, \lambda) \sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} \xi_2(\alpha_1, q, s:\ k, \gamma, \lambda) a_n \\ &\le \sum_{n=2}^{\infty} \xi_n(\alpha_1, q, s:\ k, \gamma, \lambda) a_n \\ &\le 1-\gamma \end{aligned}$$

and therefore

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\gamma}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)}. \tag{18}$$

Since $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$,

$$| f(z) | \le | z | + | z |^2 \sum_{n=2}^{\infty} a_n \, | z |^{n-2} \le r + r^2 \sum_{n=2}^{\infty} a_n \le r + \frac{(1-\gamma)}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r^2$$

which yields the right hand side of (15).
Also from Theorem (2.1), we have

$$\frac{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)}{2} \sum_{n=2}^{\infty} n a_n \le \sum_{n=2}^{\infty} \xi_n(\alpha_1, q, s:\ k, \gamma, \lambda) a_n \le 1-\gamma$$

Thus,

$$| f'(z) | \le 1 + \sum_{n=2}^{\infty} n\, a_n \, | z |^{n-1} \le 1 + r \sum_{n=2}^{\infty} n\, a_n \le 1 + \frac{2(1-\gamma)}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r.$$

On the other hand,

$$| f'(z) | \ge 1 - \sum_{n=2}^{\infty} n\, a_n \, | z |^{n-1} \ge 1 - r \sum_{n=2}^{\infty} n\, a_n \ge 1 - \frac{2(1-\gamma)}{\xi_2(\alpha_1, q, s:\ k, \gamma, \lambda)} r.$$

This completes the proof.

4 Radii of Convexity, Starlikeness and Close-to-Convexity

Theorem 4.1 *If $f \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$, then f is convex of order ρ in $|z| < R$, where*

$$R = \inf_{n\geq 2}\left[\frac{(1-\rho)\,\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{n(n-\rho)(1-\gamma)}\right]^{\frac{1}{n-1}}. \tag{19}$$

Proof. By a computation, we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} n a_n |z|^{n-1}}$$

Thus f is convex of order ρ if

$$\sum_{n=2}^{\infty}\frac{n(n-\rho)}{1-\rho}a_n |z|^{n-1} \leq 1. \tag{20}$$

Since $f \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$, we have

$$\sum_{n=2}^{\infty}\frac{\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{1-\gamma}a_n \leq 1. \tag{21}$$

Now, (21) holds if

$$\frac{n(n-\rho)}{1-\rho} |z|^{n-1} \leq \frac{\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{1-\gamma},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\,\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{n(n-\rho)(1-\gamma)}\right]^{\frac{1}{n-1}} \tag{22}$$

which yields the desired result.

Corollary 4.2 *If $f \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$, then f is starlike of order ρ in $|z| < R$, where*

$$R = \inf_{n\geq 2}\left[\frac{(1-\rho)\,\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{(n-\rho)(1-\gamma)}\right]^{\frac{1}{n-1}}. \tag{23}$$

Corollary 4.3 *If $f \in \mathcal{I}_T(\alpha_1, q, s: k, \gamma, \lambda)$, then f is close-to-convex of order ρ in $|z| < R$, where*

$$R = \inf_{n\geq 2}\left[\frac{(1-\rho)\,\xi_n(\alpha_1, q, s: k, \gamma, \lambda)}{(1-\gamma)}\right]^{\frac{1}{n-1}}. \tag{24}$$

5 Extreme points of the class $\mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$

Theorem 5.1 *Let* $f_1(z) = z$ *and* $f_n(z) = z - \dfrac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)} z^n$, $n = 2, 3, 4, \ldots$ *. Then* $f \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$ *if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n \, f_n(z), \tag{25}$$

where $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Suppose $f(z)$ can be expressed as in (25). Then

$$f(z) = \sum_{n=1}^{\infty} \lambda_n \left\{ z - \frac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)} z^n \right\} = z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \lambda_n \frac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)} \, \frac{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Thus $f \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$.

Conversely, suppose $f \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$, then $a_n \leq \dfrac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)}, \ \ldots,$ and therefore we set $\lambda_n = \dfrac{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} a_n$, $n = 2, 3, \ldots,$ and $\lambda_1 = 1 - \displaystyle\sum_{n=2}^{\infty} \lambda_n$.

Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$. Hence the proof is complete.

Corollary 5.2 *The extreme points of* $\mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$ *are the functions* $f_1(z) = z$ *and* $f_n(z) = z - \dfrac{1-\gamma}{\xi_n(\alpha_1, q, s:\ k, \gamma, \lambda)} z^n$, $n = 2, 3, 4, \ldots$

6 Integral means inequalities for fractional derivatives of $\mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$

Definition 6.1 *[12]The fractional derivative of order* λ *is defined, for a function* f, *by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \quad (0 \leq \delta < 1), \tag{26}$$

where the function f *is analytic in a simply-connected region of the complex* z*-plane containing the region and the multiplicity of* $(z-\zeta)^{-\delta}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $(z-\zeta) > 0$.

Definition 6.2 *[17]Under the hypothesis of definition 6.1, the fractional derivative of order $(m+\delta)$ is defined, for a function f, by*

$$D_z^{m+\delta} f(z) = \frac{d^m}{dz^m} D_z^{\delta} f(z)$$

where $0 \leq \delta < 1$ and $m = 0, 1, 2, \ldots$

It follows from (26) in definition 6.1 that

$$D_z^{\delta} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} z^{k-\delta} \quad (0 \leq \delta < 1). \tag{27}$$

We shall also need the following subordination theorem of Littlewood [11] in our investigation.

Lemma 6.3 *If the functions f and g are analytic in $\mathcal{U}$ with $g(z) \prec f(z)$, then, for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} \mid g(re^{i\theta}) \mid^{\tau} d\theta \leq \int_0^{2\pi} \mid f(re^{i\theta}) \mid^{\tau} d\theta$$

Theorem 6.4 *Let $f \in \mathcal{I}_T(\alpha_1, q, s:\ k, \gamma, \lambda)$ and suppose that*

$$\sum_{n=2}^{\infty} (n-m)_{m+1} a_n \leq \frac{(1-\gamma)\,\Gamma(r+1)\,\Gamma(3-m-\delta)}{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)\,\Gamma(r+1-\delta-m)\,\Gamma(2-m)} \quad (r \geq 2)$$

for $0 \leq \delta < 1$. Also let the function f_r be defined by

$$f_r(z) = z - \frac{1-\gamma}{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)} z^r,\ r = 2, 3, 4, \ldots . \tag{28}$$

If there exists an analytic function w defined by

$$\{w(z)\}^{r-1} = \frac{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} \frac{\Gamma(r+1-\delta-m)}{\Gamma(r+1)} \sum_{n=2}^{\infty} (n-m)_{m+1} \Phi(n) a_n z^{n-1}$$

with $\Phi(n) = \dfrac{\Gamma(n-m)}{\Gamma(n+1-\delta-m)}$ $\quad (0 \leq \delta < 1 \quad ; n = 2, 3, \ldots)$, then, for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \mid D_z^{m+\delta} f(z) \mid^{\tau} d\theta \leq \int_0^{2\pi} \mid D_z^{m+\delta} f_k(z) \mid^{\tau} d\theta. \quad (0 \leq \delta < 1; \tau > 0).$$

Proof. By virtue of the fractional derivative formula (27) and definition (6.2), we have

$$D_z^{m+\delta} f(z) = \frac{z^{1-m-\delta}}{\Gamma(2-\delta-m)}\left(1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\delta-m)\,\Gamma(n+1)}{\Gamma(n+1-\delta-m)}a_n z^n\right)$$

$$= \frac{z^{1-m-\delta}}{\Gamma(2-\delta-m)}\left(1-\sum_{n=2}^{\infty}\Gamma(2-\delta-m)\,(n-m)_{m+1}\Phi(n)a_n z^{n-1}\right)$$

where $\Phi(n) = \dfrac{\Gamma(n-m)}{\Gamma(n+1-\delta-m)}, \quad 0 \leq \delta < 1,\ n = 2,3,4,\dots$
Since $\Phi(n)$ is a decreasing function of n, we have,

$$0 < \Phi(n) \leq \Phi(2) = \frac{\Gamma(2-m)}{\Gamma(3-m-\delta)} \quad 0 \leq \delta < 1,\ n = 2,3,4,\dots$$

Similarly, from (27), (28) and definition 6.1, we obtain

$$D_z^{m+\delta} f_r(z) = \frac{z^{1-m-\delta}}{\Gamma(2-\delta-m)}\left(1-\frac{1-\gamma}{\xi_r(\alpha_1,q,s:\ k,\gamma,\lambda)}\,\frac{\Gamma(2-\delta-m)\,\Gamma(r+1)}{\Gamma(r+1-\delta-m)}z^{r-1}\right).$$

To prove the theorem, we must show that for $\tau > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}\Gamma(2-\delta-m)\,(n-m)_{m+1}\Phi(n)a_n z^{n-1}\right|^{\tau} d\theta$$

$$\leq \int_0^{2\pi}\left|1-\frac{1-\gamma}{\xi_r(\alpha_1,q,s:\ k,\gamma,\lambda)}\,\frac{\Gamma(2-\delta-m)\,\Gamma(r+1)}{\Gamma(r+1-\delta-m)}z^{r-1}\right|^{\tau} d\theta.$$

Thus, by applying Lemma 6.3, it would suffice to show that

$$1-\sum_{n=2}^{\infty}\Gamma(2-\delta-m)\,(n-m)_{m+1}\Phi(n)a_n z^{n-1} \tag{29}$$

$$\prec 1-\frac{1-\gamma}{\xi_r(\alpha_1,q,s:\ k,\gamma,\lambda)}\,\frac{\Gamma(2-\delta-m)\,\Gamma(r+1)}{\Gamma(r+1-\delta-m)}z^{r-1}.$$

If the subordination (29) holds true, then an analytic function w with $w(0) = 0$ and $\mid w(z) \mid < 1$ such that

$$1-\sum_{n=2}^{\infty}\Gamma(2-\delta-m)\,(n-m)_{m+1}\Phi(n)a_n z^{n-1}$$

$$= 1 - \frac{1-\gamma}{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)} \frac{\Gamma(2-\delta-m)\,\Gamma(r+1)}{\Gamma(r+1-\delta-m)} \{w(z)\}^{r-1}.$$

By the condition of the theorem, we define the function w by

$$\{w(z)\}^{r-1} = \frac{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} \frac{\Gamma(r+1-\delta-m)}{\Gamma(r+1)} \sum_{n=2}^{\infty} (n-m)_{m+1} \Phi(n) a_n z^{n-1}$$

which readily yields $w(0) = 0$. For such a function w, we have

$$\begin{aligned} | w(z) |^{r-1} &\leq \frac{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} \frac{\Gamma(r+1-\delta-m)}{\Gamma(r+1)} \sum_{n=2}^{\infty} (n-m)_{m+1} \Phi(n) a_n \mid z \mid^{n-1} \\ &\leq | z | \ \frac{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} \frac{\Gamma(r+1-\delta-m)}{\Gamma(r+1)} \Phi(2) \sum_{n=2}^{\infty} (n-m)_{m+1} a_n \\ &= | z | \ \frac{\xi_r(\alpha_1, q, s:\ k, \gamma, \lambda)}{1-\gamma} \frac{\Gamma(r+1-\delta-m)}{\Gamma(r+1)} \frac{\Gamma(2-m)}{\Gamma(3-m-\delta)} \sum_{n=2}^{\infty} (n-m)_{m+1} a_n \\ &= | z | < 1 \end{aligned}$$

by means of the hypothesis of the theorem. This means that the subordination (29) holds true, therefore the theorem is proved.

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