

Semi-Parallel Lightlike Hypersurfaces of Indefinite Sasakian manifolds

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Abstract

We study some geometric properties of a semi-parallel lightlike hypersurface M of an indefinite Sasakian manifold \overline{M} , tangent to the structure vector field. Some Theorems are obtained. We show that a semi-parallel totally contact umbilical lightlike hypersurface is totally contact geodesic.

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1 Introduction

The notion of pseudo-parallel immersions into space forms was introduced in [1], and others references therein, as an extrinsic analogue of pseudo-symmetric manifolds and also as a direct generalization of semi-parallel immersions. A hypersurface M is semi-parallel if its second fundamental form h satisfies,

$$(R(X, Y) \cdot h)(X_1, X_2) = -h(R(X, Y)X_1, X_2) - h(X_1, R(X, Y)X_2) = 0, \quad (1)$$

for any $X, Y, X_1, X_2 \in \Gamma(TM)$, where R is the curvature tensor field of M .

A $(2n + 1)$ -dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *indefinite Sasakian manifold* if it admits an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1, 1)$ of rank $2n$, ξ is a vector field, and η is a 1-form, satisfying

$$\begin{aligned} \overline{\phi}^2 &= -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0, \quad \overline{\phi}\xi = 0, \quad \eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \\ \overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) &= \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \quad (\nabla_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{\phi}\overline{X}, \overline{Y}), \\ (\nabla_{\overline{X}}\overline{\phi})\overline{Y} &= \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X}, \quad \nabla_{\overline{X}}\xi = -\overline{\phi}(\overline{X}), \quad \forall \overline{X}, \overline{Y} \in \Gamma(T\overline{M}) \end{aligned} \quad (2)$$

where $\bar{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric \bar{g} .

A plane section σ in $T_p\bar{M}$ is called a $\bar{\phi}$ -section if it is spanned by \bar{X} and $\bar{\phi}\bar{X}$, where \bar{X} is a unit tangent vector field orthogonal to ξ . The sectional curvature of a $\bar{\phi}$ -section σ is called a $\bar{\phi}$ -sectional curvature. A Sasakian manifold \bar{M} with constant $\bar{\phi}$ -sectional curvature c is said to be a *Sasakian space form* and is denoted by $\bar{M}(c)$. The curvature tensor \bar{R} of a Sasakian space form $\bar{M}(c)$ is given by [4]

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} &= \frac{c+3}{4} (\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}) + \frac{c-1}{4} (\eta(\bar{X})\eta(\bar{Z})\bar{Y} \\ &\quad - \eta(\bar{Y})\eta(\bar{Z})\bar{X} + \bar{g}(\bar{X}, \bar{Z})\eta(\bar{Y})\xi - \bar{g}(\bar{Y}, \bar{Z})\eta(\bar{X})\xi + \bar{g}(\bar{\phi}\bar{Y}, \bar{Z})\bar{\phi}\bar{X} \\ &\quad - \bar{g}(\bar{\phi}\bar{X}, \bar{Z})\bar{\phi}\bar{Y} - 2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{\phi}\bar{Z}), \quad \bar{X}, \bar{Y}, \bar{Z} \in \Gamma(T\bar{M}). \end{aligned} \tag{3}$$

Let (\bar{M}, \bar{g}) be a $(2n + 1)$ -dimensional semi-Riemannian manifold with index s , $0 < s < 2n + 1$ and let (M, g) be a hypersurface of \bar{M} , with $g = \bar{g}|_M$. M is a lightlike hypersurface of \bar{M} if g is of constant rank $2n - 1$ and the normal bundle TM^\perp is a distribution of rank 1 on M [3]. A complementary bundle of TM^\perp in TM is a rank $2n - 1$ non-degenerate distribution over M . It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As TM^\perp lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 1.1 [3] *Let $(M, g, S(TM))$ be a lightlike hypersurface of (\bar{M}, \bar{g}) . Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of TM^\perp on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on \mathcal{U} satisfying $g(N, E) = 1$ and $\bar{g}(N, N) = \bar{g}(N, W) = 0, \forall W \in \Gamma(S(TM)|_{\mathcal{U}}$.*

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(E)$ the smooth sections of the vector bundle E . Also by \perp and \oplus we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 1.1 we may write down the following decomposition

$$\begin{aligned} TM &= S(TM) \perp TM^\perp, \\ T\bar{M} &= TM \oplus N(TM) = S(TM) \perp (TM^\perp \oplus N(TM)). \end{aligned} \tag{4}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on (\bar{M}, \bar{g}) , then by using the second decomposition of (4) and considering a normalizing pair $\{E, N\}$ as in Theorem 1.1, we have Gauss and Weingarten formulae in the form

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{5}$$

$$\text{and } \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N = -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{6}$$

where $\nabla_X Y, A_N X \in \Gamma(TM)$. ∇ is an induced a symmetric linear connection on M , ∇^\perp is a linear connection on the vector bundle $N(TM)$, B is a symmetric bilinear form and A_N is the shape operator of M concerning N .

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (5), we obtain $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, E)$ and $\tau(X) = \bar{g}(\nabla_X^\perp N, E)$.

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM . We have

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E \quad \text{and} \quad \nabla_X E = -A_E^* X - \tau(X)E, \quad (7)$$

where $\nabla_X^* PY$ and $A_E^* X$ belong to $\Gamma(S(TM))$. C, A_E^* and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. Note that ∇ is not a metric connection. Also, we have the following identities, $g(A_E^* X, PY) = B(X, PY)$, $g(A_E^* X, N) = 0$, $B(X, E) = 0$.

Finally, using (5), \bar{R} and R are the curvature tensor fields of \bar{M} and M are related as

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ ((\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z))N \end{aligned} \quad (8)$$

2 Results

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface, *tangent to the structure vector field* ξ ($\xi \in TM$). If E is a local section of TM^\perp , then $\bar{g}(\bar{\phi}E, E) = 0$, and $\bar{\phi}E$ is tangent to M . Thus $\bar{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\bar{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\bar{\phi}(TM^\perp)$ as vector subbundle. If we consider a local section N of $N(TM)$, since $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$, we deduce that $\bar{\phi}N$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\bar{g}(\bar{\phi}N, N) = 0$, we see that the component of $\bar{\phi}N$ with respect to E vanishes. Thus $\bar{\phi}N \in \Gamma(S(TM))$. From (2), we have $\bar{g}(\bar{\phi}N, \bar{\phi}E) = 1$. Therefore, $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

It is known [2] that if M is tangent to the structure vector field ξ , then, ξ belongs to $S(TM)$. Using this, and since $\bar{g}(\bar{\phi}E, \xi) = \bar{g}(\bar{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n - 4$ on M such that

$$S(TM) = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))\} \perp D_0 \perp \langle \xi \rangle, \quad (9)$$

where $\langle \xi \rangle$ is the distribution spanned by ξ , that is, $\langle \xi \rangle = \text{Span}\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\bar{\phi}$, i.e. $\bar{\phi}(D_0) = D_0$.

Moreover, from (4) and (9) we obtain the decomposition

$$TM = \{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\} \perp D_0 \perp \langle \xi \rangle \perp TM^\perp, \quad (10)$$

$$T\overline{M} = \{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\} \perp D_0 \perp \langle \xi \rangle \perp (TM^\perp \oplus N(TM)) \quad (11)$$

Now, we consider the distributions on M , $D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle \xi \rangle. \quad (12)$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (12), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D' , respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to X and (2) (note that $\overline{\phi}^2 N = -N$), we obtain $\overline{\phi}X = \phi X + u(X)N$, where ϕ is a tensor field of type $(1, 1)$ defined on M by $\phi X := \overline{\phi}RX$.

Next, we investigate the effect of semi-parallel condition on the geometry of lightlike hypersurfaces in an indefinite Sasakian space form.

Theorem 2.1 *Let M be a semi-parallel lightlike hypersurface of an indefinite Sasakian space $\overline{M}(c)$ of constant curvature c , with $\xi \in TM$. Then either $c = 1$ or M is $(\overline{\phi}(TM^\perp), D \oplus D')$ -mixed totally geodesic. Moreover, if $c = 1$, then either M is totally geodesic or $C(E, A_E^*PX) = 0$, for any $X \in \Gamma(TM)$.*

Proof: Using (3), (5), (8) into (1) and after calculation, we obtain

$$\begin{aligned} & \frac{c+3}{4} (g(Y, X_1)B(X, X_2) - g(X, X_1)B(Y, X_2)) \\ & + \frac{c-1}{4} (\eta(X)\eta(X_1)B(Y, X_2) - \eta(Y)\eta(X_1)B(X, X_2) - \eta(Y)u(X_2)g(X, X_1) \\ & + \eta(X)u(X_2)g(Y, X_1) + \overline{g}(\overline{\phi}Y, X_1)B(\phi X, X_2) - \overline{g}(\overline{\phi}X, X_1)B(\phi Y, X_2) \\ & - 2\overline{g}(\overline{\phi}X, Y)B(\phi X_1, X_2)) - B(X, X_1)B(A_N Y, X_2) + B(Y, X_1)B(A_N X, X_2) \\ & + \frac{c+3}{4} (g(Y, X_2)B(X, X_1) - g(X, X_2)B(Y, X_1)) \\ & + \frac{c-1}{4} (\eta(X)\eta(X_2)B(Y, X_1) - \eta(Y)\eta(X_2)B(X, X_1) \\ & - \eta(Y)u(X_1)g(X, X_2) + \eta(X)u(X_1)g(Y, X_2) + \overline{g}(\overline{\phi}Y, X_2)B(\phi X, X_1) \\ & - \overline{g}(\overline{\phi}X, X_2)B(\phi Y, X_1) - 2\overline{g}(\overline{\phi}X, Y)B(\phi X_2, X_1)) \\ & - B(X, X_2)B(A_N Y, X_1) + B(Y, X_2)B(A_N X, X_1) = 0. \end{aligned} \quad (13)$$

Then, by taking $X = E$ into (13), with the aid of $B(E, \cdot) = 0$, we have

$$\frac{c-1}{4} (\overline{g}(\overline{\phi}Y, X_1)B(\phi E, X_2) + u(X_1)B(\phi Y, X_2) + 2u(Y)B(\phi X_1, X_2))$$

$$\begin{aligned}
& + B(Y, X_1)B(A_N E, X_2) + \frac{c-1}{4} (\bar{g}(\bar{\phi}Y, X_2)B(\phi E, X_1) + u(X_2)B(\phi Y, X_1) \\
& + 2u(Y)B(\phi X_2, X_1)) + B(Y, X_2)B(A_N E, X_1) = 0. \tag{14}
\end{aligned}$$

Again, by taking $X_2 = E$ into (14), we get $\frac{3}{4}(c-1)u(Y)B(V, X_1) = 0$. Putting $Y = U$ into this equation, we derive $\frac{3}{4}(c-1)B(V, X_1) = 0$. Now, if $B(V, X_1) \neq 0, \forall X_1 \in \Gamma(D \oplus D')$, then $c = 1$. If $c \neq 1$, then $B(V, X_1) = 0, \forall X_1 \in \Gamma(D \oplus D')$, that is, M is $(\bar{\phi}(TM^\perp), D \oplus D')$ -mixed totally geodesic. On the other hand, suppose that $c = 1$. From (14) and taking $X_1 = X_2$, we obtain, $B(Y, X_1)B(A_N E, X_1) = 0$. If $B(Y, X_1) = 0, \forall Y, X_1 \in \Gamma(TM)$, then M is totally geodesic. If $B(Y, X_1) \neq 0$, then $B(A_N E, X_1) = 0$, that is, $C(E, A_E^* P X_1) = 0$, for any $X_1 \in \Gamma(TM)$. \square

It is easy to check that the condition $C(E, A_E^* P X) = 0$, for any $X \in \Gamma(TM)$, of the Theorem 2.1 is related with a part of components of the Ricci tensor Ric of M . That is $0 = C(E, A_E^* P X) = R(E, X)$. From the Theorem 2.2 in [3], page 88, we have the following characterization on the lightlike geometry of M .

Corollary 2.2 (to Theorem 2.1) *Let $(M, g, S(TM))$ be a semi-parallel lightlike hypersurface of an indefinite Sasakian space $\bar{M}(c = 1)$ of constant curvature $c = 1$, with $\xi \in TM$, such that $Ric(E, X) \neq 0$, for any $X \in \Gamma(TM)$ and $E \in \Gamma(TM^\perp)$. Then*

- (i) $A_W^* X = 0$, for any $W \in \Gamma(TM^\perp)$ and $X \in \Gamma(TM)$,
- (ii) The connection ∇ induced by $\bar{\nabla}$ on M is torsion-free and metric,
- (iii) TM^\perp is a parallel distribution with respect to ∇ ,
- (iv) TM^\perp is a Killing distribution on M .

A submanifold M is said to be totally contact umbilical lightlike hypersurface of the a semi-Riemannian manifold \bar{M} if the second fundamental form h of M satisfies [4]

$$h(X, Y) = (g(X, Y) - \eta(X)\eta(Y))H - (\eta(X)u(Y) + \eta(Y)u(X))N, \tag{15}$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$).

A totally contact umbilical submanifold which has nonzero parallel mean curvature vector (i.e. $\nabla^\perp H = 0$) is called an *extrinsic sphere*.

Theorem 2.3 *Let $\bar{M}(c)$ be an indefinite Sasakian space form and M be a totally contact umbilical lightlike hypersurface of $\bar{M}(c)$ with $\xi \in TM$. Then $c = -3$ ($\bar{M}(c)$ is of constant curvature -3) and M cannot be an extrinsic sphere.*

Proof: The first assertion follows from Theorem 2.4 in [4]. We also have, for any $X \in \Gamma(TM)$, $\nabla_E^\perp H = \bar{g}(H, E)^2 N$ and $\nabla_{P_X}^\perp H = 0$. So, $\nabla_X^\perp H \neq 0$, for any $X \in \Gamma(TM)$ which completes the proof. \square

If M is a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$ with $\xi \in TM$, we have $\bar{g}((R(E, Y) \cdot h)(X_1, E), E) = \frac{3}{4}(c - 1)\lambda u(Y)u(X_1)$. If the second fundamental form h of lightlike hypersurface M satisfies (1), then, we have $0 = \bar{g}((R(E, Y) \cdot h)(X_1, E), E) = \frac{3}{4}(c - 1)\lambda u(Y)u(X_1)$ which leads, by taking $Y = X_1 = U$, to $\frac{3}{4}(c - 1)\lambda = 0$. By Theorem 2.3, $c = -3 \neq 1$, then $\lambda = 0$ and $h(X, Y) = -(\eta(X)u(Y) + \eta(Y)u(X))N$, $\forall X, Y \in \Gamma(TM)$. So, M is totally contact geodesic. Therefore we have

Theorem 2.4 *Let (M, g) be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian manifold (\bar{M}, \bar{g}) , with $\xi \in TM$. If the second fundamental form h of M satisfies (1), then M is totally contact geodesic.*

It is well known that the geometry of a lightlike hypersurface depends on the chosen screen distribution. As the local second fundamental form B of M on \mathcal{U} is independent of the choice of the screen distribution [3], all results of this paper are stable with respect to any change of the screen distribution.

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