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Multiplication Operators Induced by Operator Valued Maps

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Abstract

In this paper we characterize multiplication operators induced by operator valued measurable functions and obtain some criteria for compact and fredholm multiplication operators.

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Let H be a Hilbert space and (X, s, μ) be a measure space. For $1 \leq p < \infty$, let $L^p(X, H) = \{f : f : X \to H$ is measurable and $\int_X ||f(.)||^p d\mu < \infty\}$. Then $L^p(X, H)$ is a Banach space under the norm,

$||f|| = (\int_X ||f(.)||^p d\mu)^{\frac{1}{p}}$ and for $p = 2, L^2(X, H)$ is a Hilbert space under the inner product,

$$< f, g >= \int_X < f(.), g(.) > d\mu$$

let $w : X \to B(H)$ be an operator valued measurable function. Then a bounded linear transformation, $M_w : L^p(X, H) \to L^p(X, H)$ defined by $(M_w f)(x) =$

$(w.f)(x) = w(x)(f(x))$ is called a multiplication operator induced by the operator valued map w.

Recently several authors have studied multiplication and weighted composition operators on different function spaces. For example, one can refer to [1], [2], [4] and [5]. For more details see [3]. In this paper we generalize the results of Takagi H and Yokouchi K [5]. In fact we characterize multiplication operators induced by operator-valued measurable functions. We also obtain some criteria for compact and Fredhlom multiplication operators induced by operator-valued maps.

Theorem 1. Suppose $1 \leq p < \infty$, and $w : X \to B(H)$ is an operator valued measurable function. Then $M_w : L^p(X, H) \to L^p(X, H)$ is a bounded operator if and only if $w \in L^\infty(X, B(H))$. Moreover, $||M_w|| = ||w||_\infty$.

Proof. If $w \in L^\infty(X, B(H))$, then clearly

$$\begin{aligned} ||M_w f||_p^p &= \int ||w(x)f(x)||^p d\mu(x) \\ &\leq \int ||w(x)||^p ||f(x)||^p d\mu(x) \\ &\leq ||w||_\infty^p ||f||_p^p \qquad \cdots (i) \end{aligned}$$

which implies that M_w is a bounded operator.

Conversly, suppose M_w is a bounded operator we show that $w \in L^\infty(X, B(H))$. For, if this is not the case, then for every positive number δ, such that the set

$$A = \{x : ||w(x)|| > \delta\} \text{ has a positive measure .}$$

We can easily find a subset B of A such that $0 < \mu(B) < \infty$ and such that for every $x \in B$ there exist a unit vector $e_x \in H$ such that $||w(x)e_x|| > \delta$. Taking

$$f = \frac{\chi_B g}{\sqrt[p]{\mu}(B)}, \text{ where } g(x) = e_x.$$

for all $x \in B$. We see that,

$$\begin{aligned} ||f||_p^p &= \frac{1}{\mu(B)} \int_B ||g(x)||^p d\mu \\ &= \frac{1}{\mu(B)} . \mu(B) = 1 \end{aligned}$$

but

$$\begin{aligned} ||M_w f||_p^p &= \int ||w(x)f(x)||^p d\mu \\ &= \frac{1}{\mu(B)} \int_B ||w(x)e_x||^p d\mu \end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{\mu(B)}\delta^p \mu(B) \\
&= \delta^p.
\end{aligned}
$$

This is true for every $\delta > 0$, which contradicts the continuity of M_w. Hence w must be essentially bounded. We now show that $||M_w|| = ||w||_\infty$. For any $\epsilon > 0$, let

$$E = \{x \in X : ||w(x)|| > (||w||_\infty - \epsilon)\}.$$

Suppose E has a positive measure. We can easily find a measurable subset F of E such that $0 < \mu(F) < \infty$. Now, for every $x \in F$, there exists $e_x \in H$ such that $||e_x|| = 1$ and $||w(x)e_x|| > ||w(x)||_\infty - \epsilon$. Let $f = \chi_F e$, where $e : X \to H$ is defined by $e(x) = e_x$. Now

$$(\chi_F e)(x) = \begin{cases} \frac{e_x}{\sqrt[p]{\mu(F)}}, & \text{if } x \in F \\ 0, & \text{if } x \notin F, \end{cases}$$

and

$$
\begin{aligned}
||M_w|| &\geq ||M_w f||_p^p \\
&= \int_F ||w(x)f(x)||^p d\mu(x) \\
&\geq (||w(x)||_\infty - \epsilon)^p
\end{aligned}
$$

Hence $||M_w|| \geq ||w||_\infty - \epsilon$. Since ϵ is arbitrary so, $||M_w|| \geq ||w||_\infty$.
From equation (i), we have

$$||M_w|| \leq ||w||_\infty$$

Hence $||M_w|| = ||w||_\infty$, which completes the proof.

Theorem 2. For $1 \leq p < \infty$, let $M_w \in C(L^p(X, H))$. Then M_w is a compact operator if and only if the space

$$E_\epsilon = L^p(X/w(X,\epsilon), H) = \{f \in L^p(X, H) : f(x) = 0 \text{ for all } x \notin w(X, \epsilon)\}$$

is a finite dimensional for each $\epsilon > 0$, where $w(X, \epsilon) = \{x \in X : ||w(x)g|| \geq \epsilon||g||, \text{ for every } g \in H\}$.

Proof. It is easy to see that the subspace E_ϵ is invariant under M_w. If M_w is a compact operator, then its restriction M_w/E_ϵ to E_ϵ is also compact operator. Moreover $M_w/E_\epsilon : E_\epsilon \to E_\epsilon$ has closed range contained in E_ϵ. Hence E_ϵ is a finite dimensional space, since a closed subspace contained in the range of a compact operator must be finite dimensional. This proves the direct part of the theorem.

To prove the converse, let E_ϵ be finite dimensional for each $\epsilon > 0$. In particular $E_{\frac{1}{n}}$ is a finite dimensional space for each $n \in N$. Define

$$w_n(x) = \begin{cases} w(x), & \text{if } x \in w(X, \frac{1}{n}) \\ 0, & \text{if } x \notin w(X, \frac{1}{n}), \end{cases}$$

Then

$$\begin{aligned} ||(M_{w_n} - M_w)f||_p^p &= \int ||((w_n(x) - w(x))f(x)||^p d\mu \\ &= \int_{w(x,\frac{1}{n})'} ||w(x)f(x)||^p d\mu \\ &= \leq \int_{w(x,\frac{1}{n})'} \frac{1}{n^p}||f(x)||^p d\mu \\ &\leq \frac{1}{n^p}||f||_p^p \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $M_{w_n} \to M_w$. But each M_{w_n} is a finite rank operator. Therefore M_w must be a compact as the limit of the finite rank operators is a compact operator.

Theorem 3. Let (X, s, μ) be a non-atomic measure space. Then no non-zero multiplication operator on $L^p(X, H)$ into itself is a compact operator.

Proof. If possible, suppose $w \neq 0$ almost everywhere, then the set $w(X, \epsilon) = \{x \in X : ||w(x)g|| \geq \epsilon||g||, \text{ for all } g \in H\}$ has positive measure. Since $w(X, \epsilon)$ is a finite set, does not contain any atoms, so there exists a decreasing sequence $\{E_n\}$ of subsets of $w(X, \epsilon)$ such that $0 < \mu(E_n) < \frac{1}{n}$ and the set $\{\chi_{E_n} : n \in N\}$ spans an infinite dimensional subspace of E_ϵ and therefore E_ϵ is infinite dimensional, which contradicts the theorem.

Theorem 4. Let (X, s, μ) be a atomic measure space. Then M_w is compact if and only if for each $\epsilon > 0$, the set $w(X, \epsilon) = \{x : ||w(x)g|| \geq \epsilon||g|| \text{ for all } g \in H\}$ is a finite set.

Proof. We first take M_w to be a compact operator and prove that the set $w(X, \epsilon) = \{x : ||w(x)g|| \geq \epsilon||g||, \text{ for all } g \in H\}$ is a finite set. For, if the set $w(X, \epsilon)$ is an infinite set, then let $\{x_k\}$ be an infinite sequence of atoms in $w(X, \epsilon)$. For any $k \in N$, consider

$$f^k(.) = \frac{\chi_{\{x_k\}}(.)e_k}{\sqrt[p]{\mu}(x_k)}$$

Then $\{f^k\}$ is an infinite sequence in the closed unit ball of $L^p(X, H)$. Now

$$||M_w f^k - M_w f^j|| = \int_X ||w(x)f^k(x) - w(x)f^j(x)||d\mu$$

$$\geq \epsilon$$

which contradicts the compactness of M_w.
Conversely, if the set

$$w(X, \frac{1}{n}) = \{x \in X : ||w(x)|| \geq \frac{1}{n}\} \text{ is finite }.$$

Let $w_n = w/w(X, \frac{1}{n})$. Then clearly M_{w_n} is a finite rank operator. Moreovere, $||M_{w_n} - M_w|| \to 0$ as $n \to \infty$. Hence M_w is a compact operator.

Theorem 5. Let $M_w \in C(L^p(X, H))$ and $S = \{x \in H : w(x) \neq 0\}$. Then M_w has closed range if and only if for every measurable subset $E \in S$, there exists $\epsilon > 0$ such that $||w(x)g|| \geq \epsilon||g||$ for $\mu-$almost all $x \in E$ and for all $g \in H$.

Proof. Suppose M_w has closed range. If the condition is false, then for every $n \in N$, we can find a measurable set E_n of X with $0 < \mu(E_n) < 1$ and a vector $e_n \in H$ such that

$$||w(x)e_n|| < \frac{1}{2^n}||e_n||$$

without loss of generality we can assume that $\{E_n\}$ is a pairwise disjoint sequence of measurable sets. Let

$$g = \sum \frac{w.\chi_{E_n} e_n}{||e_n|| \sqrt[p]{\mu}(E_n)}.$$

then

$$\begin{aligned}
\int ||g(x)||^p d\mu &= \sum_{n=1}^{\infty} \int_{E_n} ||w(x)e_n||^p d\mu \\
&< \sum_{n=1}^{\infty} \frac{1}{\mu(E_n)} \int_{E_n} |\frac{1}{2^n}|^p d\mu \\
&< \sum_{n=1}^{\infty} \frac{1}{2^{np}} \\
&< \infty
\end{aligned}$$

Set

$$f_n = \sum_{k=1}^{n} \frac{\chi_{E_k}(.)e_k}{\sqrt[p]{\mu}(E_k)||e_k||}$$

Then

$$\begin{aligned}
||M_w f_n - g||_p^p &= \int ||w(x)f_n(x) - g(x)||^p d\mu \\
&= \sum_{k>n} \frac{1}{\mu(E_k)} \int_{E_k} ||w(x_k)\frac{e_k}{||e_k||}||^p d\mu \\
&< \sum_{k>n} \frac{1}{2^{kp}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $g \in \overline{ranM_w}$. Therefore $g = M_w f$ for some $f \in L^p(X,H)$. For $x \in E_n$, we have $g(x) = w(x)f(x)$ which implies that $f(x) = \dfrac{e_n}{\sqrt[p]{\mu}(E_n)||e_n||}$ for all $x \in E_n$ This contradicts the fact that $f \in L^p(X,H)$. Hence the condition must hold. Conversely, if the condition is true, then

$$\begin{aligned} ||M_w f||_p^p &\geq \int_S ||w(x)f(x)||^p d\mu(x) \\ &\geq \epsilon \int_S ||f(x)||^p d\mu(x) \qquad \cdots (i) \end{aligned}$$

Suppose $M_w f^{(n)} \to g$ for some sequence $\{f^n\} \subset L^p(X,H)$. Clearly

$$||M_w f^n - M_w f^m|| \geq \epsilon \int_S ||f^n - f^m||^p d\mu.$$

Now sequence $\{f^n\}$ is a Cauchy sequence in $L^p(X,H)$, where

$$\hat{f}^n(x) = \begin{cases} f^n(x), & \text{if } x \in S \\ 0, & \text{if } x \notin S, \end{cases}$$

Therefore, there exists $f \in L^p(X,H)$ such that $||\hat{f}^n - f||_p \to 0$ as $n \to \infty$, so that $M_w \hat{f}^n \to M_w f$ or equivalently $M_w f^n \to M_w f$, so that $g = M_w f$. This proves that M_w has closed range.

Theorem 6. Let (X, s, μ) be a non-atomic measure space. Let $M_w \in C(L^2(X,H))$. Then M_w is Fredholm if and only if M_w is invertible.

Proof. If M_w is Fredholm, then for every measurable subset E of S, there exists $\epsilon > 0$ such that $||w(x)g|| \geq \epsilon||g||$ for all $x \in E$. if $\mu(X \setminus E) > 0$, then clearly M_w has finite dimensional kernel. Keeping this in mind we take X = E. For any $g \in H$, $||w(x)w^{-1}(x)g|| \geq ||w^{-1}(x)g||$ for all $x \in E$ implies that $M_{w^{-1}}$ is a bounded operator. Clearly $M_{w^{-1}}$ is the inverse of M_w. Hence M_w is a invertible operator.

The converse part of the theorem is trivial.

Theorem 7. Let (X, s, μ) be a $\sigma-$finite atomic measure space. Then $M_w : L^2(X,H) \to L^2(X,H)$ is a Fredholm operator if and only if

(i) for every measurable subset E of supp w there exists $\epsilon > 0$ such that $||w(x)g|| \geq \epsilon||g||$ for all $x \in E$ and $g \in H$.

(ii) $w(x)$ is one-to-one for all but finitely many $x \in X$.

Proof. Suppose M_w is Fredholm. In view of closed range of M_w the condition (i) follows from Theorem 6. We shall only prove the condition (ii). To prove this let $w(x_k)$ be not one-to-one for infinitely many $x_k \in X$. We can easily find $g_k \in H$ such that $g_k \in kerw(x_k)$. Now $\chi_{\{x_k\}} \in kerM_w$ for infinitely k. So that $kerM_w$ becomes infinite diemnsional, which is a contradiction. Hence $w(x)$ is injective for all but finitely many $x \in X$.

The proof of the converse part is trivial.

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