

# Solving Fuzzy Nonlinear Equations in Banach Spaces

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## Abstract

In this paper, we suggest and analyze a new two-step iterative method for solving nonlinear fuzzy equations using the Midpoint quadrature rule. The fuzzy quantities are presented in parametric form. Several examples are given to illustrate the efficiency of the proposed method.

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**Keywords:** Fuzzy nonlinear equations; Fixed point iteration; Newton's method

## 1 Introduction

In recent years much attention has been given to develop iterative type methods for solving nonlinear equations like  $F(x) = 0$ . The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by Zadeh [13]. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers [1, 6, 10]. Standard analytical techniques presented by Buckley and Qu in [2 - 4], cannot be suitable for solving the equations such as

$$\begin{aligned} (i) \quad & ax^5 + bx^4 + cx^3 + dx^2 + ex + f = g, \\ (ii) \quad & x - \sin(x) = g, \end{aligned}$$

where  $x, a, b, c, d, e, f$  and  $g$  are fuzzy numbers. In this paper we have an adjustment on the classic Newton's method in order to accelerate the convergence or to reduce the number of operations and evaluations in each step of the iterative process. We suggest and analyze an iterative method by using the Midpoint rule. This method is an implicit-type method. To implement

this, we use Newton's method as predictor method and then use this method as corrector method. Several examples are given to illustrate the efficiency and advantage of this two-step method. In Section 2, we bring some basic definitions and results on fuzzy numbers. In Section 3 we introduce Midpoint Newton's method for solving of nonlinear real equations. In Section 4 we apply introduced method from Section 3 for solving of nonlinear fuzzy equations. The proposed algorithm is illustrated by some examples in Section 5, and conclusion is in Section 6.

## 2 Preliminaries

**Definition 2.1** A fuzzy number is set like  $u : \mathbb{R} \rightarrow I = [0, 1]$  which satisfies, [8, 12, 14],

1.  $u$  is upper semi-continuous,
2.  $u(x) = 0$  outside some interval  $[c, d]$ ,
3. There are real numbers  $a, b$  such that  $c \leq a \leq b \leq d$  and
  - 3.1.  $u(x)$  is monotonic increasing on  $[c, a]$ ,
  - 3.2.  $u(x)$  is monotonic decreasing on  $[a, b]$ ,
  - 3.3.  $u(x) = 1, a \leq x \leq b$ .

**Definition 2.2** A fuzzy number  $u$  in parametric form is a pair  $(\underline{u}, \bar{u})$  of functions  $\underline{u}(r), \bar{u}(r), 0 \leq r \leq 1$ , which satisfies the following requirements:

1.  $\underline{u}(r)$  is a bounded monotonic increasing left continuous function,
2.  $\bar{u}(r)$  is a bounded monotonic decreasing left continuous function,
3.  $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$ .

A crisp number  $\alpha$  is simply represented by  $\underline{u}(r) = \bar{u}(r) = \alpha, 0 \leq r \leq 1$ .

A popular fuzzy number is triangular fuzzy number  $u = (a, b, c)$ , with the membership function

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{x-c}{b-c}, & b \leq x \leq c, \end{cases}$$

where  $c \neq a, c \neq b$  and hence

$$\underline{u}(r) = a + (c - a)r, \quad \bar{u}(r) = b + (c - b)r.$$

Let  $TF(\mathbb{R})$  be the set of all triangular fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary  $u = (\underline{u}, \bar{u})$ ,  $v = (\underline{v}, \bar{v})$  and  $k > 0$  we defined addition  $u + v$  and multiplication by real number  $k > 0$  as

$$\begin{aligned} (\underline{u + v})(r) &= \underline{u}(r) + \underline{v}(r), & (\overline{u + v})(r) &= \bar{u}(r) + \bar{v}(r), \\ (\underline{ku})(r) &= k\underline{u}(r), & (\overline{ku})(r) &= k\bar{u}(r). \end{aligned}$$

### 3 Midpoint Newton’s method

We consider the problem of finding a real zero of a function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ , that is, a real solution  $\alpha$ , of the nonlinear equation system  $F(x) = 0$ , of  $n$  equations with  $n$  variables. This solution can be obtained as a fixed point of some function  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by means of the fixed point iteration method

$$x_{k+1} = G(x_k), \quad k = 0, 1, \dots,$$

where  $x_0$  is the initial estimation. The best known fixed point method is the classical Newton’s method, given by

$$x_{k+1} = x_k - J_F(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \dots,$$

where  $J_F(x_k)$  is the Jaccobian Matrix of the function  $F$  evaluated in  $x_k$ .

Let  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a sufficiently differentiable function and  $\alpha$  be a zero of the system of nonlinear equations  $F(x) = 0$ . The following result will be used describe the Newton’s method and Midpoint Newton’s method; see its proof in[9].

**Lemma 3.1** *Let  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuously differentiable on a convex set  $D$ . Then, for any  $x, y \in D$ ,  $F$  satisfies*

$$F(y) - F(x) = \int_0^1 J_F(x + t(y - x))(y - x)dt. \tag{1}$$

Once the iterate  $x_k$  has been obtained, using (1):

$$F(y) = F(x_k) + \int_0^1 J_F(x_k + t(y - x_k))(y - x_k)dt. \tag{2}$$

If we estimate  $J_F(x_k + t(y - x_k))$  in the interval  $[0, 1]$  by its value in  $t = 0$ , that is by  $J_F(x_k)$ , and take  $y = \alpha$ , then

$$0 \approx F(x_k) + J_F(x_k)(\alpha - x_k),$$

is obtained, and a new approximation of  $\alpha$  can be done by

$$x_{k+1} = x_k - J_F(x_k)^{-1}F(x_k),$$

what is the classical *Newton method* (CN) for  $k = 0, 1, \dots$

If an estimation of (2) is made by means of the Midpoint rule and  $y = \alpha$  is taken, then

$$0 \approx F(x_k) + J_F\left(\frac{x_k + \alpha}{2}\right)(\alpha - x_k),$$

is obtained and a new approximation  $x_{k+1}$  of  $\alpha$  is given by

$$x_{k+1} = x_k - J_F\left(\frac{x_k + x_{k+1}}{2}\right)^{-1}F(x_k).$$

In order to avoid the implicit problem that this equation involves, we use the  $(k + 1)$ th iteration of Newton method in the right side. Then,

$$x_{k+1} = x_k - J_F\left(\frac{x_k + z_k}{2}\right)^{-1}F(x_k), \quad k = 0, 1, \dots, \quad (3)$$

where

$$z_k = x_k - J_F(x_k)^{-1}F(x_k).$$

This method is called *Midpoint Newton's method* (MN).

The Midpoint Newton's method can be understood as a *substitution of  $J_F(x_k)$  in Newton's method by  $J_F(\frac{x_k+z_k}{2})$* .

## 4 Midpoint Newton's method for fuzzy equations

Now our aim in this section is to obtain a solution for nonlinear equation  $F(x) = 0$ .

The parametric form is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \bar{x}; r) = 0, \\ \overline{F}(\underline{x}, \bar{x}; r) = 0. \end{cases} \quad \forall r \in [0, 1] \quad (4)$$

Suppose that  $\alpha = (\underline{\alpha}, \bar{\alpha})$  is the solution to the system (4), i.e.,

$$\begin{cases} \underline{F}(\underline{\alpha}, \bar{\alpha}; r) = 0, \\ \overline{F}(\underline{\alpha}, \bar{\alpha}; r) = 0, \end{cases} \quad \forall r \in [0, 1]$$

Therefore, if  $x_0 = (\underline{x}_0, \bar{x}_0)$  is an approximation solution for this system, then  $\forall r \in [0, 1]$ , there are  $h(r), k(r)$  such that

$$\begin{cases} \underline{\alpha}(r) = \underline{x}_0(r) + h(r), \\ \bar{\alpha}(r) = \bar{x}_0(r) + k(r). \end{cases}$$

Now by using of the Taylor series of  $\underline{F}, \overline{F}$  about  $(\underline{x}_0, \overline{x}_0)$ , then  $\forall r \in [0, 1]$ ,

$$\begin{cases} \underline{F}(\underline{\alpha}, \overline{\alpha}; r) = \underline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) + O(\Gamma) = 0, \\ \overline{F}(\underline{\alpha}, \overline{\alpha}; r) = \overline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) + O(\Gamma) = 0, \end{cases}$$

where  $\Gamma = h^2 + hk + k^2$  and if  $\underline{x}_0$  and  $\overline{x}_0$  are near to  $\underline{\alpha}$  and  $\overline{\alpha}$ , respectively, then  $h(r)$  and  $k(r)$  are small enough. Let us suppose that all needed partial derivatives exists are bounded. Therefore for enough small  $h(r)$  and  $k(r)$ , where  $\forall r \in [0, 1]$ , we have,

$$\begin{cases} \underline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) = 0, \\ \overline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) = 0, \end{cases}$$

and hence  $h(r)$  and  $k(r)$  are unknown quantities that can be obtained by solving the following equations,  $\forall r \in [0, 1]$ ,

$$J_F(\underline{x}_0, \overline{x}_0; r) \begin{bmatrix} h(r) \\ k(r) \end{bmatrix} = \begin{bmatrix} -\underline{F}(\underline{x}_0, \overline{x}_0; r) \\ -\overline{F}(\underline{x}_0, \overline{x}_0; r) \end{bmatrix}, \tag{5}$$

where

$$J_F(\underline{x}_0, \overline{x}_0; r) = \begin{bmatrix} \underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) & \underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) \\ \overline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) & \overline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) \end{bmatrix}.$$

is the Jaccobian Matrix of the function  $F = (\underline{F}, \overline{F})$  evaluated in  $x_0 = (\underline{x}_0, \overline{x}_0)$ . Hence, the next approximations for  $\underline{x}(r)$  and  $\overline{x}(r)$  are as follows

$$\begin{cases} \underline{x}_1(r) = \underline{x}_0(r) + h(r), \\ \overline{x}_1(r) = \overline{x}_0(r) + k(r), \end{cases}$$

for all  $r \in [0, 1]$ .

We can obtain approximated solution,  $r \in [0, 1]$ , by using the recursive scheme

$$\begin{cases} \underline{x}_{n+1}(r) = \underline{x}_n(r) + h_n(r), \\ \overline{x}_{n+1}(r) = \overline{x}_n(r) + k_n(r), \end{cases} \tag{6}$$

where  $n = 1, 2, \dots$  Analogous to (5),

$$J_F(\underline{x}_n, \overline{x}_n; r) \begin{bmatrix} h_n(r) \\ k_n(r) \end{bmatrix} = \begin{bmatrix} -\underline{F}(\underline{x}_n, \overline{x}_n; r) \\ -\overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}, \tag{7}$$

Now, let  $J_F(\underline{x}_n, \overline{x}_n; r)$  be nonsingular, then from (6) recursive scheme of Newton's method is obtained as follows,

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - J_F(\underline{x}_n, \overline{x}_n; r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n; r) \\ \overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}. \tag{8}$$

From *Midpoint Newton's method* (MN) in Section 3, by substitution of  $J_F(\underline{x}_n, \bar{x}_n; r)$  in (7) by  $J_F\left(\frac{(\underline{x}_n, \bar{x}_n; r) + (\underline{z}_n, \bar{z}_n; r)}{2}\right)$ , where

$$\begin{bmatrix} \underline{z}_n \\ \bar{z}_n \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \bar{x}_n(r) \end{bmatrix} - J_F(\underline{x}_n, \bar{x}_n; r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \bar{x}_n; r) \\ \bar{F}(\underline{x}_n, \bar{x}_n; r) \end{bmatrix},$$

then similar to (8) in Newton's method, recursive scheme for Midpoint Newton's method is obtained as follows

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \bar{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \bar{x}_n(r) \end{bmatrix} - J_F\left(\frac{(\underline{x}_n, \bar{x}_n; r) + (\underline{z}_n, \bar{z}_n; r)}{2}\right)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \bar{x}_n; r) \\ \bar{F}(\underline{x}_n, \bar{x}_n; r) \end{bmatrix}, \tag{9}$$

where  $n = 1, 2, \dots$ . For initial guess, one can use the fuzzy number

$$x_0 = (\underline{x}(0), \underline{x}(1), \bar{x}(0))$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \quad \bar{x}_0(r) = \bar{x}(0) + (\underline{x}(1) - \bar{x}(0))r,$$

when  $\underline{x}(0) \leq \underline{x}(1) \leq \bar{x}(0)$ .

**Remark 1.** Sequence  $\{(\underline{x}_n, \bar{x}_n)\}_{n=0}^\infty$  convergent to  $(\underline{\alpha}, \bar{\alpha})$  iff  $\forall r \in [0, 1]$ ,  $\lim_{n \rightarrow \infty} \underline{x}_n(r) = \underline{\alpha}(r)$  and  $\lim_{n \rightarrow \infty} \bar{x}_n(r) = \bar{\alpha}(r)$ .

**Lemma 4.1** *Let  $F(\underline{\alpha}, \bar{\alpha}) = (\underline{F}(\underline{\alpha}, \bar{\alpha}), \bar{F}(\underline{\alpha}, \bar{\alpha})) = (0, 0)$  and if the sequence  $\{(\underline{x}_n, \bar{x}_n)\}_{n=0}^\infty$  converges to  $(\underline{\alpha}, \bar{\alpha})$  according to Midpoint Newton's method, then*

$$\lim_{n \rightarrow \infty} P_n = 0,$$

where

$$P_n = \sup_{0 \leq r \leq 1} \max\{h_n(r), k_n(r)\}.$$

**Proof.** For  $\forall r \in [0, 1]$  in convergent case we have

$$\lim_{n \rightarrow \infty} h_n(r) = \lim_{n \rightarrow \infty} k_n(r) = 0,$$

which completes the proof. □

Finally, in the following it is shown that, under certain conditions, Midpoint Newton's method for fuzzy equation  $F(x) = 0$  is convergent and that this convergence is quadratical.

**Theorem 4.1** *Let  $\forall r \in [0, 1]$ , the functions  $\underline{F}$  and  $\bar{F}$  are continuously differentiable with respect to  $\underline{x}$  and  $\bar{x}$ . Assume that there exists  $(\underline{\alpha}(r), \bar{\alpha}(r)) \in \mathbb{R}^2$  and a  $\beta > 0$  such that  $\|J_F(\underline{\alpha}, \bar{\alpha}; r)^{-1}\| \leq \beta$  and  $J_F$  will be Lipschitz continuous with respect to  $\underline{x}$  and  $\bar{x}$  with constant  $\gamma$ , then the Midpoint Newton's method converges to  $(\underline{\alpha}, \bar{\alpha})$ , and there exists a  $M > 0$  such that,*

$$\|(\underline{x}_{n+1}, \bar{x}_{n+1}) - (\underline{\alpha}, \bar{\alpha})\| \leq M \|(\underline{x}_n, \bar{x}_n) - (\underline{\alpha}, \bar{\alpha})\|^2.$$

**Proof.** From [5], for  $n = 2$ , the result is concluded. □

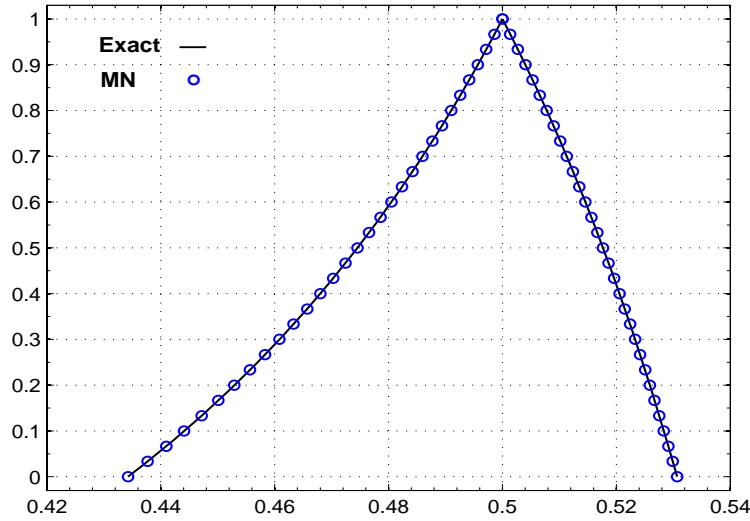


Figure 1: Solution of example 1

## 5 Numerical application

In this section we will check the effectiveness of Midpoint Newton’s method.

**Example 5.1** Consider the fuzzy nonlinear equation[2]

$$(3, 4, 5)x^2 + (1, 2, 3)x = (1, 2, 3).$$

Without any loss of generality, assume that  $x$  is positive, then the parametric form of this equation is as follows

$$\begin{cases} (3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) = (1 + r), \\ (5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) = (3 - r), \end{cases}$$

or equality

$$\begin{cases} (3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) - (1 + r) = 0, \\ (5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) - (3 - r) = 0. \end{cases}$$

To obtain initial guess we use above system for  $r = 0$  and  $r = 1$ , therefore

$$\begin{cases} 3\underline{x}^2(0) + \underline{x}(0) - 1 = 0, \\ 5\overline{x}^2(0) + 3\overline{x}(0) - 3 = 0, \end{cases} \quad \text{and} \quad \begin{cases} 4\underline{x}^2(1) + 2\underline{x}(1) - 2 = 0, \\ 4\overline{x}^2(1) + 2\overline{x}(1) - 2 = 0. \end{cases}$$

Consequently  $\underline{x}(0) = 0.43425, \overline{x}(0) = 0.53066$  and  $\underline{x}(1) = \overline{x}(1) = 0.5$ . Therefore initial guess is  $x_0 = (0.43425, 0.5, 0.53066)$ . After two iterations, we obtain the solution by Midpoint Newton’s method with the maximum error less than  $10^{-11}$ , and by classical Newton’s method after two iterations the

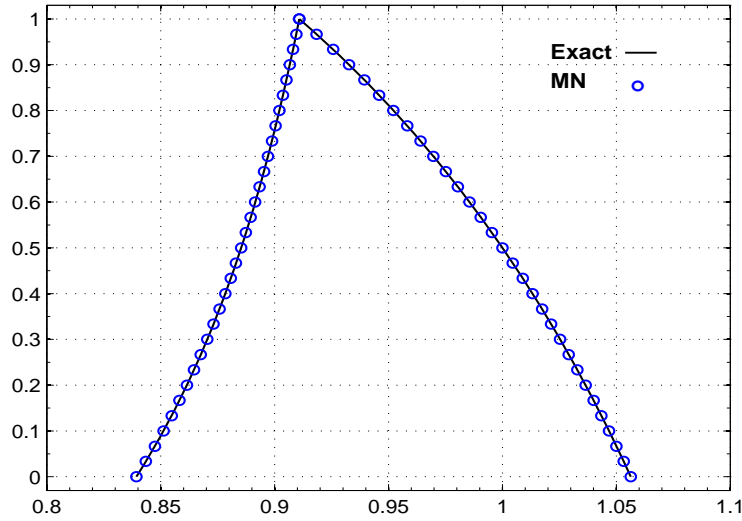


Figure 2: Positive solution of example 2

maximum error would be less than  $10^{-5}$ . For more details see Figure 1. Now suppose  $x$  is negative, we have

$$\begin{cases} (3 + r)\bar{x}^2(r) + (1 + r)\underline{x}(r) - (1 + r) = 0, \\ (5 - r)\underline{x}^2(r) + (3 - r)\bar{x}(r) - (3 - r) = 0. \end{cases}$$

For  $r = 0$ , we have,  $\underline{x}(0) \simeq -0.90705$  and  $\bar{x}(0) \simeq -1.11373$ , hence  $\underline{x}(0) > \bar{x}(0)$ , therefore negative root does not exist.

**Example 5.2** Consider fuzzy nonlinear equation[2]

$$(1, 2, 3)x^3 + (2, 3, 4)x^2 + (3, 4, 5) = (5, 8, 13).$$

Without any loss of generality, assume that  $x$  is positive, then parametric form of this equation is as follows

$$\begin{cases} (1 + r)\underline{x}^3(r) + (2 + r)\underline{x}^2(r) + (3 + r) = (5 + 3r), \\ (3 - r)\bar{x}^3(r) + (4 - r)\bar{x}^2(r) + (5 - r) = (13 - 5r), \end{cases}$$

or equality

$$\begin{cases} (1 + r)\underline{x}^3(r) + (2 + r)\underline{x}^2(r) - (2 + 2r) = 0, \\ (3 - r)\bar{x}^3(r) + (4 - r)\bar{x}^2(r) - (8 - 4r) = 0, \end{cases}$$

Similar to Example 5.1, to obtain initial guess we use above system for  $r = 0$  and  $r = 1$ , therefore

$$\begin{cases} \underline{x}^3(0) + 2\underline{x}^2(0) - 2 = 0, \\ 3\bar{x}^3(0) + 4\bar{x}^2(0) - 8 = 0, \end{cases} \quad \text{and} \quad \begin{cases} 2\underline{x}^3(1) + 3\underline{x}^2(1) - 4 = 0, \\ 2\bar{x}^3(1) + 3\bar{x}^2(1) - 4 = 0. \end{cases}$$

Consequently  $\underline{x}(0) = 0.83928, \bar{x}(0) = 1.05636$  and  $\underline{x}(1) = \bar{x}(1) = 0.91082$ . Therefore initial guess is  $x_0 = (0.83928, 0.91082, 1.05636)$ . After two iterations, we obtain the solution by Midpoint Newton's method with the maximum error less than  $10^{-8}$ , and by classical Newton's method after two iterations the maximum error would be less than  $10^{-3}$ . For more details see Figure 2.

## 6 Conclusion

In this paper, we suggested numerical solving method for fuzzy nonlinear equations. This method is an implicit-type method. To implement this, we use Newton's method as predictor method and then use this method as corrector method. The method is discussed in detail. Several examples are given to illustrate the efficiency and advantage of this two-step method.

## References

- [1] R. Badard, The law of large numbers for fuzzy processes and the estimation problem, *Inform. Sci.* **28** (1982) 161-178.
- [2] J.J. Buckley, Y. Qu, solving linear and quadratic fuzzy equation, *Fuzzy Set Syst.* **38** (1990) 43-59.
- [3] J.J. Buckley, Y. Qu, On using  $\alpha$ -cuts to evaluate equations, *Fuzzy Sets Syst.* **38** (1990) 309-312.
- [4] J.J. Buckley, Y. Qu, Solving fuzzy equations: A new solution concept, *Fuzzy Set Syst.* **39** (1991) 291-301.
- [5] A. Corder, J.R. Torregrosa, Variant of Newton's method for functions of several variables, *Appl. Math. Comput.* **183** (2006), 199-208.
- [6] P. Diamond, Fuzzy least squares, *Inform. Sci.* **46** (1988) 144-157.
- [7] J.E. Dennis, R.B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, New Jersey, 1983.
- [8] D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Application*, Academic Press, new York, 1980.
- [9] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, Inc., 1970.
- [10] K. Peeva, Fuzzy linear systems, *Fuzzy Sets Syst.* **49** (1992) 339-355.

- [11] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, second ed., Springer-Verlag, 1993.
- [12] L.A. Zadeh, Fuzzy sets, Inform. Contr. **8** (1965) 338-353.
- [13] L.A. Zadeh, The concept of a linguistic variable and its application to approximation reasoning, Inform. Sci. **3** (1975) 199-249.
- [14] H.J. Zimmermann, Fuzzy Sets Theory and Its Application, Kluwer Academic Press, Dordrecht, 1991.

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