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Notes on Maximal Ideals Relative to a Filter[1]

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Abstract

In this paper, an intrinsic characterization of distributive lattices is obtained. In addition, we also give a characterization of pseudo primes in semicontinuous lattices.

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1 Introduction and Preliminaries

The study of semiprime ideals was begun in [5] by Y. Ray. The theory of semicontinuous lattices was first developed by D. Zhao in [1]. In this paper, we manage to give an intrinsic characterizations of distributive lattices. A characterization of pseudo primes in semicontinuous lattices is also obtained.

The following are some basic concepts needed in the sequel, other non-explicitly stated elementary notions please refer to [1], [3] and [6].

An *ideal* on a partially ordered set (in short, *poset*) L means a lower set which is also directed, and a *filter* on a poset can be dually defined. For a semilattice L, a proper ideal I of L is called a *prime ideal* if for any two elements a, b of L, $a \wedge b \in I$ implies $a \in I$ or $b \in I$. For a lattice L, an ideal I of L is called a *semiprime ideal* if for any three elements a, b, c of L, the relations $a \wedge b \in I, a \wedge c \in I$ always imply $a \wedge (b \vee c) \in I$. The set of semiprime

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ideals of L is denoted by $Rd(L)$. It is easy to see that every prime ideal is semiprime in a lattice.

Recall that in a complete lattice L, for $x, y \in L$, we say that $x \Leftarrow y$, if for any $I \in Rd(L)$, $y \leq \bigvee I$ always implies $x \in I$. For any $x \in L$, let $\Downarrow x = \{y \in L : y \Leftarrow x\}$ and $\Uparrow x = \{y \in L : x \Leftarrow y\}$. A complete lattice L is said to be *semicontinuous lattice* if for each $x \in L$, $x \leq \bigvee \Downarrow x$.

2 Main Results

In this section, we shall give an intrinsic characterization of distributive lattices. A characterization of pseudo primes in the case of semicontinuous lattices will be obtained.

Lemma 2.1. (see [4]) *Let L be a sup-semilattice. Let $I \in Idl\ L$, $F \in Filt\ L$ and $I \cap F = \emptyset$. Then I is a maximal ideal relative to filter F iff for all $x \in L \backslash I$, there are $y \in F$ and $a \in I$ s.t. $y \leq x \vee a$.*

Theorem 2.2. *Let M be a semiprime ideal on a lattice. If M is a maximal ideal relative to a filter, then M is a prime ideal.*

Proof. Let L be a lattice. Let M be a maximal ideal relative to filter F and $M \cap F = \emptyset$. Suppose there are $a, b \in M$ s.t. $a \wedge b \in M$ but $a \notin M$ and $b \notin M$. By Lemma 2.1, there are $u, v \in F$, $c, d \in M$ s.t. $u \leq a \vee c$, $v \leq b \vee d$, respectively. Since F is a filter, $a \vee c$, $b \vee d \in F$ and $(a \vee c) \wedge (b \vee d) \in F$. Noticing that $c, d \in M$ and M is a semiprime ideal, we have $a \wedge b \in M$, $a \wedge d \in M$ and $a \wedge (b \vee d) \in M$; and also $c \wedge b \in M$, $c \wedge d \in M$ and $c \wedge (b \vee d) \in M$. It follows from M is a semiprime ideal that $(a \vee c) \wedge (b \vee d) \in M$. This shows that $(a \vee c) \wedge (b \vee d) \in M \cap F \neq \emptyset$, a contradiction! Hence, M is a prime ideal. □

This theorem shows that the similar result may be obtained in the non-distributive case. An example is given by Figure 1 to show that maximal ideals relative to a filter may be prime ideals in the non-distributive case, where $I = \{a, b, c, d, 0\}$ is a maximal ideal relative to filter $L \backslash I$ and a prime ideal but L is not a distributive lattice.

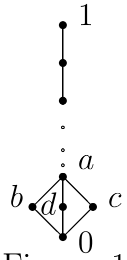

Figure 1

By Figure 1, we find that there exists a maximal ideal $\downarrow b$ relative to filter $\uparrow a$ but not a semiprime ideal in the non-distributive lattice L.

Lemma 2.3. (see [4]) *Let L be a poset, $I \in Idl\ L$, $F \in Filt\ L$ and $I \cap F = \emptyset$. Then there always exists a maximal ideal M relative to filter F s.t. $M \cap F = \emptyset$ and $M \supseteq I$.*

Theorem 2.4. *If maximal ideals relative to a filter on a lattice L are all semiprime ideals, then L is a distributive lattice.*

Proof. Let L be a lattice and $a, b, c \in L$. Let $x = a \wedge (b \vee c)$ and $y = (a \wedge b) \vee (a \wedge c)$. It is trivial that $x \geq y$. On the other hand, suppose that $x \not\leq y$. Then $\uparrow x \bigcap \downarrow y = \emptyset$. By Lemma 2.3, there exists a maximal ideal M relative to filter $\uparrow x$ s.t. $M \cap \uparrow x = \emptyset$ and $M \supseteq \downarrow y$. Thus $x \notin M$ and $y \in M$. Since M is a semiprime ideal, $a \wedge b \in M, a \wedge c \in M$ and $x = a \wedge (b \vee c) \in M$, a contradiction! Hence $x \leq y$, and thus L is distributive. □

Corollary 2.5. *If maximal ideals on a lattice L are all semiprime ideals, then L is a distributive lattice.*

By Theorem 2.4, Corollary 2.5 and the fact that maximal ideals relative to a filter on a distributive lattice are all prime ideals(see [4]), we have an intrinsic characterization of distributive lattices

Theorem 2.6. *Let L be a lattice. Then the following conditions are equivalent:*

(1) *L is a distributive lattice;*
(2) *Maximal ideals relative to a filter on L are all prime ideals;*
(3) *Maximal ideals relative to a filter on L are all semiprime ideals;*
(4) *Maximal ideals on L are all semiprime ideals;*
(5) *Maximal ideals on L are all prime ideals.*

Recall that an element p of a poset L is called *pseudo prime element* if $p = \bigvee P$ for some prime ideal P. All the pseudo prime elements of L is denoted by ψPRIME L.

Now we give the following characterization of pseudo primes in semicontinuous lattices.

Lemma 2.7. *([3]) Let L be a distributive lattice, I an ideal and F a filter in L with $I \cap F = \emptyset$. Then there is a prime ideal P in L with $P \supseteq I$ and $P \cap F = \emptyset$.*

Theorem 2.8. *Let L be a complete lattice and $1 \neq p \in L$. Consider the following statements:*

(1) *p is pseudo prime;*
(2) *In any finite collection $x_1, x_2, \cdots, x_n \in L$ with $x_1 \wedge x_2 \wedge \cdots \wedge x_n \Leftarrow p$ there is one of the elements with $x_j \leq p$*
(3) *The filter generated by $L \backslash \downarrow p$ does not meet $\Downarrow p$.*

Then (1)$\Rightarrow$ (2) and (2)$\Leftrightarrow$ (3) ; if L is in addition distributive semicontinuous, all three statements are equivalent.

Proof. Condition (2) says that no finite meet of elements from $L \backslash {\downarrow} p$ is ever $\Leftarrow p$. Therefore (2) and (3) are always equivalent.

(1) implies (2): Let p be pseudo prime and suppose that $x_1 \wedge x_2 \wedge \cdots \wedge x_n \Leftarrow p$. Let P be a prime ideal with $\bigvee P = p$. Since every prime ideal is semiprime, $P \in Rd(L)$, thus $x_1 \cdots x_n \in P$. Since P is prime, there is one $j \in \{1, 2, \cdots, n\}$ with $x_j \in P \subseteq \downarrow p$. That is, $x_j \leq p$.

(3) implies (1): Suppose that L is semicontinuous. Let F be the filter generated by $L \backslash {\downarrow} p$, then $L \backslash {\downarrow} p \subseteq F$ and $F \cap \Downarrow p = \emptyset$. By Lemma 2.7, there exists a prime ideal P with $P \supseteq \Downarrow p$ and $P \cap F = \emptyset$. Since that $L \backslash {\downarrow} p \subseteq F$, we have $P \subseteq L \backslash F \subseteq {\downarrow} p$. Since L is semicontinuous, $p \leq \bigvee \Downarrow p \leq \bigvee P \leq \bigvee {\downarrow} p = p$. Thus $p = \bigvee P$ is pseudo prime. □

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