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Properties of Harmonic Functions which are Starlike of Complex Order with Repect to Symmetric Points

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Abstract

Let $\mathcal{H}$ denote the class of functions f which are harmonic, orientation preserving and univalent in the open unit disc $D = \{z : |z| < 1\}$. This paper defines and investigates a family of complex-valued harmonic functions that are orientation preserving and univalent in $\mathcal{D}$ and are related to the functions starlike of complex order with respect to symmetric points. The authors obtain extreme points, convolution and convex combination properties.

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1 Introduction

Let $f = u+iv$ be a continuous complex-valued harmonic function in a complex domain E if both u and v are real harmonic in the domain E. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re\ (U)$ and $v = Im\ (V)$. Then, we can write

$$f(z) = h(z) + \overline{g(z)}$$

where h and g are analytic in E. The mapping $z \mapsto f(z)$ is orientation preserving and locally univalent in E if and only if the Jacobian of f given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive in E. The function $f = h + \overline{g}$ is said to be harmonic univalent in E if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one-to-one in E. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let $\mathcal{H}$ denote the family of functions $f = h + \overline{g}$ that are harmonic, orientation preserving and univalent in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ with the normalisation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

In [3], Clunie and Sheil-Small investigated the class $\mathcal{H}$ plus some of it geometric subclasses and obtained some coefficient bounds. Since then, there have been many authors which looked at related subclasses. See [8] and [9] to name a few. In particular, Jahangiri [4] discussed a subclass of $\mathcal{H}$ consisting of functions which are starlike of α, for $0 \leq \alpha < 1$. We denote such class as $\mathcal{HS}^\star(\alpha)$. Specifically, a function f of the form (1) is harmonic starlike of order α, $0 \leq \alpha < 1$, for $z \in \mathcal{D}$ if (see Sheil-Small [6])

$$\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) \geq \alpha, \quad |z| = r < 1.$$

Next, we denote further the class $\overline{\mathcal{H}}$, a subclass of $\mathcal{H}$ such that the functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{2}$$

Also let $\overline{\mathcal{HS}}^\star(\alpha) = \mathcal{HS}^\star(\alpha) \cap \overline{H}$.

In [5], Nasr and Aouf introduced the class of starlike functions of complex order b. Denote $\mathcal{S}^*(b)$ to be the class consisting of functions which are analytic and starlike of complex order b(b is a non-zero complex number) and satisfying the following condition

$$Re\ \left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0,\ z \in \mathcal{D}.$$

In [1], Janteng and Abdul Halim were motivated to form a new subclass of $\mathcal{H}$ based on Nasr and Aouf's class as follows.

Definition 1.1 *Let $f \in \mathcal{H}$. Then $f \in \mathcal{HS}_s^\star(b,\alpha)$ is said to be harmonic starlike of complex order, with respect to symmetric points, if and only if, for $0 \leq \alpha < 1$, b non-zero complex number with $|b| \leq 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$,*

$$Re\ \left\{1 + \frac{1}{b}\left(\frac{2zf'(z)}{z'(f(z) - f(-z))} - 1\right)\right\} \geq \alpha,\ |z| = r < 1.$$

Also, we let $\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha) = \mathcal{HS}_s^\star(b,\alpha) \cap \overline{H}$. The constraint $|b| \leq 1$ is to ensure $J_f(z) > 0$ so that f is univalent.

2 Main Results

Avci and Zlotkiewicz [2] proved that the coefficient condition $\sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right) \leq 1$ is a sufficient condition for functions $f = h+\overline{g}$ to be in $\mathcal{HS}^\star(1,0)$ with $b_1 = 0$. Silverman [7] also proved that this condition is also a necessary when a_n and b_n are negative, as well as $b_1 = 0$. In the following theorem, Jahangiri in 1999 [4], obtained analogue sufficient condition for $f \in \mathcal{HS}^\star(1,\alpha)$ where b_1 is not necessarily 0.

Theorem 2.1 *([4]) Let $f = h + \overline{g}$ be given by (1). Furthermore, let*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 1.$$

where $a_1 = 1$ and $0 \leq \alpha < 1$. Then f is harmonic univalent in $\mathcal{D}$, and $f \in \mathcal{HS}^\star(1,\alpha)$.

Jahangiri also proved that the condition in Theorem 2.1 is a necessary condition for $f = h + \overline{g}$ given by (2) and belongs to $\overline{\mathcal{H}}\mathcal{S}^\star(1,\alpha)$.

The following theorem proved by Janteng and Abdul Halim in [1] will be used throughout in this paper.

Theorem 2.2 *Let $f = h + \overline{g}$ be given by (1). If*

$$\sum_{n=2}^{\infty}\left(\frac{2n + (|b| - \alpha|b| - 1)(1 - (-1)^n)}{2(1-\alpha)|b|}\right)|a_n| + \sum_{n=1}^{\infty}\left(\frac{2n - (|b| - \alpha|b| - 1)(1 - (-1)^n)}{2(1-\alpha)|b|}\right)|b_n| \leq 1, \tag{3}$$

where $0 \leq \alpha < 1$ and b a non-zero complex number with $|b| \leq 1$ then f is harmonic univalent in $\mathcal{D}$, and $f \in \mathcal{HS}_s^\star(b,\alpha)$. Condition (3) is also necessary if $f \in \overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$.

Next, extreme points of the closed convex hulls of $\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$ are determined, and denoted by $clco\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$.

Theorem 2.3 *$f \in clco\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \tag{4}$$

where

$$h_1(z) = z,\ h_n(z) = z - \left(\frac{2(1-\alpha)|b|}{2n + (|b| - \alpha|b| - 1)(1-(-1)^n)}\right)\ z^n\ (n = 2, 3, ...),$$

$$g_n(z) = z + \left(\frac{2(1-\alpha)|b|}{2n - (|b| - \alpha|b| - 1)(1-(-1)^n)}\right)\ (\bar{z})^{\ n}\ (n = 1, 2, 3, ...),$$

$\sum_{n=1}^{\infty}(X_n + Y_n) = 1,\ X_n \geq 0$ and $Y_n \geq 0$. In particular, the extreme points of $\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f having the form (4), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \\ &= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{2(1-\alpha)|b|}{2n + (|b| - \alpha|b| - 1)(1-(-1)^n)}X_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{2(1-\alpha)|b|}{2n - (|b| - \alpha|b| - 1)(1-(-1)^n)}Y_n \bar{z}^{\ n} \end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}\frac{2n + (|b| - \alpha|b| - 1)(1-(-1)^n)}{2(1-\alpha)|b|}\left(\frac{2(1-\alpha)|b|}{2n + (|b| - \alpha|b| - 1)(1-(-1)^n)}\right)X_n$$

$$+\sum_{n=1}^{\infty}\frac{2n - (|b| - \alpha|b| - 1)(1-(-1)^n)}{2(1-\alpha)|b|}\left(\frac{2(1-\alpha)|b|}{2n - (|b| - \alpha|b| - 1)(1-(-1)^n)}\right)Y_n$$

$$\begin{aligned} &= \sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n \\ &= 1 - X_1 \\ &\leq 1. \end{aligned}$$

Therefore $f \in clco\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$.

On the converse, we suppose $f \in clco\overline{\mathcal{H}}\mathcal{S}_s^\star(b,\alpha)$. Set

$$X_n = \frac{2n + (|b| - \alpha|b| - 1)(1-(-1)^n)}{2(1-\alpha)|b|}|a_n|\ , (n = 2, 3, 4, ...),$$

and

$$Y_n = \frac{2n - (|b| - \alpha|b| - 1)(1 - (-1)^n)}{2(1 - \alpha)|b|}|b_n| \ , (n = 1, 2, 3, ...),$$

From Theorem 2.2, we can deduce that $0 \le X_n \le 1$,$(n = 2, 3, 4, ...)$ and $0 \le Y_n \le 1$,$(n = 1, 2, 3, ...)$. We define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$. Again from Theorem 2.2, $X_1 \ge 0$. Therefore, $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ as required.

For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^{\ n}$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \bar{z}^{\ n}$, we define the convolution of f and F as

$$(f \star F)(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \bar{z}^{\ n}. \tag{5}$$

In the next theorem, we examine the convolution properties of the class $\overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha)$.

Theorem 2.4 *For $0 \le \beta \le \alpha < 1$, let $f \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha)$ and $F \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \beta)$. Then $(f \star F) \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha) \subset \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \beta)$.*

Proof. Write $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^{\ n}$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \bar{z}^{\ n}$. Then the convolution of f and F is given by (5).

Note that $|A_n| \le 1$ and $|B_n| \le 1$ since $F \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \beta)$. Then we have

$$\sum_{n=2}^{\infty} [2n + (|b| - \alpha|b| - 1)(1 - (-1)^n)]|a_n||A_n| + \sum_{n=1}^{\infty} [2n - (|b| - \alpha|b| - 1)(1 - (-1)^n)]|b_n||B_n|$$

$$\le \sum_{n=2}^{\infty} [2n + (|b| - \alpha|b| - 1)(1 - (-1)^n)]|a_n| + \sum_{n=1}^{\infty} [2n - (|b| - \alpha|b| - 1)(1 - (-1)^n)]|b_n|.$$

Therefore, $(f \star F) \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha) \subset \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \beta)$ since the right hand side of the above inequality is bounded by $2(1 - \alpha)$ while $2(1 - \alpha) \le 2(1 - \beta)$. $\square$

Now, we determine the convex combination properties of the members of $\overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha)$.

Theorem 2.5 *The class $\overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i \in \overline{\mathcal{H}\mathcal{S}}_s^{\star}(b, \alpha)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^{\ n}.$$

For $\sum_{i=1}^{\infty} c_i = 1, \ 0 \le c_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} c_i f_i(z) \quad = \quad c_1 z - \sum_{n=2}^{\infty} c_1 |a_{n,1}| z^n + \sum_{n=1}^{\infty} c_1 |b_{n,1}| \bar{z}^{\ n} + c_2 z - \sum_{n=2}^{\infty} c_2 |a_{n,2}| z^n + \sum_{n=1}^{\infty} c_2 |b_{n,2}| \bar{z}^{\ n} ...$$

$$\begin{aligned} &= z\sum_{i=1}^{\infty} c_i - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|a_{n,i}|\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} c_i|b_{n,i}|\right)\bar{z}^{\,n} \\ &= z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|a_{n,i}|\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} c_i|b_{n,i}|\right)\bar{z}^{\,n}. \end{aligned}$$

Next, consider

$$\begin{aligned} &\sum_{n=2}^{\infty}\left([2n+(|b|-\alpha|b|-1)(1-(-1)^n)]\left|\sum_{i=1}^{\infty} c_i|a_{n,i}|\right|\right) \\ &+\sum_{n=1}^{\infty}\left([2n-(|b|-\alpha|b|-1)(1-(-1)^n)]\left|\sum_{i=1}^{\infty} c_i|b_{n,i}|\right|\right) \\ &= c_1\sum_{n=2}^{\infty}[2n+(|b|-\alpha|b|-1)(1-(-1)^n)]|a_{n,1}| + ... \\ &\quad + c_m\sum_{n=2}^{\infty}[2n+(|b|-\alpha|b|-1)(1-(-1)^n)]|a_{n,m}| + ... \\ &\quad + c_1\sum_{n=1}^{\infty}[2n-(|b|-\alpha|b|-1)(1-(-1)^n)]|b_{n,1}| + ... \\ &\quad + c_m\sum_{n=1}^{\infty}[2n-(|b|-\alpha|b|-1)(1-(-1)^n)]|b_{n,m}| + ... \\ &= \sum_{i=1}^{\infty} c_i\Big\{\sum_{n=2}^{\infty}[2n+(|b|-\alpha|b|-1)(1-(-1)^n)]|a_{n,i}| \\ &\quad + \sum_{n=1}^{\infty}[2n-(|b|-\alpha|b|-1)(1-(-1)^n)]|b_{n,i}|\Big\}. \end{aligned}$$

Now, $f_i \in \overline{\mathcal{HS}}_s^{\star}(b,\alpha)$, therefore from Theorem 2.2, we have

$$\sum_{n=2}^{\infty}[2n+(|b|-\alpha|b|-1)(1-(-1)^n)]|a_{n,i}|+\sum_{n=1}^{\infty}[2n-(|b|-\alpha|b|-1)(1-(-1)^n)]|b_{n,i}| \le 2(1-\alpha)\ .$$

Hence

$\sum_{n=2}^{\infty}\left([2n+(|b|-\alpha|b|-1)(1-(-1)^n)]\left|\sum_{i=1}^{\infty} c_i|a_{n,i}|\right|\right)$

$+\sum_{n=1}^{\infty}\left([2n-(|b|-\alpha|b|-1)(1-(-1)^n)]\left|\sum_{i=1}^{\infty} c_i|b_{n,i}|\right|\right)$

$\le 2(1-\alpha)\sum_{i=1}^{\infty} c_i$

$= 2(1-\alpha).$

By using Theorem 2.2 again, we have $\sum_{i=1}^{\infty} c_i f_i \in \overline{\mathcal{HS}}_s^{\star}(b,\alpha)$. $\square$

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