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On Total Dominating Sets in Graphs

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Abstract

A set S of vertices in a graph $G(V,E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A set S of vertices in a graph $G(V,E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of S. The domination number of a graph G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in G. Respectively the total domination number of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. An upper bound for $\gamma_t(G)$ which has been achieved by Cockayne and et al. in [1] is: for any graph G with no isolated vertex and maximum degree $\Delta(G)$ and n vertices, $\gamma_t(G) \leq n - \Delta(G) + 1$.
Here we characterize bipartite graphs and trees which achieve this upper bound. Further we present some another upper and lower bounds for $\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

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1 Introduction

Let $G(V,E)$ be a graph. For any vertex $x \in V$, we define the neighborhood of x, denoted by $N(x)$, as the set of all vertices adjacent to x. The closed neighborhood of x, denoted by $N[x]$, is the set $N(x) \cup \{x\}$. For a set of vertices S, we define $N(S)$ as the union of $N(x)$ for all $x \in S$, and $N[S] = N(S) \cup S$. The degree of a vertex is the size of its neighborhoods. The maximum degree of a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. Here n will denote the number of vertices of a graph G. A set S of vertices in a graph $G(V,E)$ is called a dominating set if every vertex $v \in V$ is either

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an element of S or is adjacent to an element of S. A set S of vertices in a graph $G(V, E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of S. The domination number of a graph G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in G. Respectively the total domination number of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. clearly $\gamma(G) \leq \gamma_t(G)$, also it has been proved that $\gamma_t(G) \leq 2\gamma(G)$.
An upper bound for $\gamma_t(G)$ has been achieved by Cockayne and et al. in [1] in the following theorems:

THEOREM A *If a graph G has no isolated vertices, then $\gamma_t(G) \leq n - \Delta(G) + 1$.*

THEOREM B *If G is a connected graph and $\Delta(G) < n-1$, then $\gamma_t(G) \leq n - \Delta(G)$*

As a result of the above theorems, if G is a graph with $\gamma_t(G) = n - \Delta(G) + 1$, then $\Delta(G) \geq n - 1$. Hence, if G is a $k-$ regular graph and $\gamma_t(G) = n - k + 1$, then G is K_n. As a result of the above theorems, if G is a graph with $\gamma_t(G) = n - \Delta(G) + 1$, then $\Delta(G) \geq n - 1$. Hence, if G is a $k-$ regular graph and $\gamma_t(G) = n - k + 1$, then G is K_n. Total domination and upper bounds on the total domination number in graphs were intensively investigated, see e. g. ([3], [4]).
Here we characterize bipartite graphs and trees which achieve the upper bound in Theorem A. Further we present some another upper and lower bounds for $\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

It is easy to prove that for $n \geq 3$, $\gamma_t(C_n) = \gamma_t(P_n) = \frac{n}{2}$ if $n \equiv 0 \pmod 4$ and $\gamma_t(C_n) = \gamma_t(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ otherwise.

for the definitions and notations not defined here we refer the reader to texts, such as [2].

2 Other bounds for $\gamma_t(G)$

In this section we introduce some other upper bounds for $\gamma_t(G)$.

Theorem 2.1 *Let G be a connected graph, then $\gamma_t(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$.*

Proof: Let $S \subseteq V(G)$ be a total dominating set in G. Every vertex in S dominates at most $\Delta(G) - 1$ vertices of $V(G) - S$ and dominate at least one of the vertices in S. Hence, $|S|(\Delta(G) - 1) + |S| \geq n$. Since, S is an arbitrary total dominating set, then $\gamma_t(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$. ■

If $G = K_n$, $G = C_{4n}$, or $G = P_{4n}$ then $\gamma_t(G) = \lceil \frac{n}{\Delta(G)} \rceil$. so the above bound is sharp.

Theorem 2.2 *Let G be a graph with $diam(G) = 2$ then, $\gamma_t(G) \leq \delta(G) + 1$.*

Proof: Let $x \in V(G)$ and $\deg(x) = \delta(G)$. Since, $\text{diam}(G) = 2$, then $N(x)$ is a dominating set for G.
Now $S = N(x) \cup \{x\}$ is a total dominating set for G and $|S| = \delta(G) + 1$. Hence, $\gamma_t(G) \leq \delta(G) + 1$. ■

As we know, $\gamma_t(C_5) = 3$ and also $\delta(C_5) = 2$, $diam(C_5) = 2$ then $\gamma_t(C_5) = \delta(C_5) + 1$. Hence, the above bound is sharp.

Theorem 2.3 *If G is a connected graph with the girth of length $g(G) \geq 5$ and $\delta(G) \geq 2$, then $\gamma_t(G) \leq n - \lceil \frac{g(G)}{2} \rceil + 1$.*

Proof: Let G be a connected graph with $g(G) \geq 5$ and let C be a cycle of length $g(G)$. Remove C from G to form a graph G'. Suppose an arbitrary vertex $v \in V(G')$, since $\delta(G) \geq 2$, then v has at least two neighbors say x and y. Let $x, y \in C$. If $d(x, y) \geq 3$, then replacing the path from x to y on C with the path x, v, y reduces the girth of G, a contradiction. If $d(x, y) \leq 2$, then x, y, v are on either C_3 or C_4 in G, contradicting the hypothesis that $g(G) \geq 5$. Hence, no vertex in G' has two or more neighbors on C. Since $\delta(G) \geq 2$, the graph G' has minimum degree at least $\delta(G) - 1 \geq 1$. Then G' has no isolated vertex. Now let S' be a γ_t–set for C. Then $S = S' \cup V(G')$ is a total dominating set for G. Hence, $\gamma_t(G) \leq n - \lceil \frac{g(G)}{2} \rceil + 1$(note that $\gamma_t(C) \leq \lfloor \frac{g(G)}{2} \rfloor + 1$) . ■

3 Bipartite graphs with $\gamma_t(G) = n - \Delta(G) + 1$

In this section we charactrize the bipartite graphs achieving the upper bound in the theorem A.

Theorem 3.4 *Let G be a bipartite graph with no isolated vertices. Then $\gamma_t(G) = n - \Delta(G) + 1$ if and only if G is a graph in form of $K_{1,t} \bigcup rK_2$ for $r \geq 0$.*

Proof: If G is $K_{1,t} \cup rK_2 (r \geq 0)$, clearly $\gamma_t(G) = n - \Delta(G) + 1$. Now let G be a bipartite graph with partitions $A \bigcup B$ and $x \in A$ where $\deg(x) = \Delta(G) = t$. We continue our proof in four stages:
Stage 1: We claim that for every vertex $y \in A - \{x\}$, $N(y) - N(x) \neq \emptyset$. If it is not true, there exists a vertex in $A - \{x\}$, say y, such that $N(y) \subseteq N(x)$. So let $u \in N(y)$, the set $S = V - (N(x) \cup \{y\}) \bigcup \{u\}$ is a total dominating set and $|S| = n - \Delta(G)$, a contradiction. So we have $n \geq 2|A| + \Delta(G) - 1$.
Stage 2: For every vertex $y \in A$, let $u_y \in N(y)$. Clearly the set $S = A \cup (\cup_{y \in A} \{u_y\})$ is a total dominating set for G and $|S| \leq 2|A|$, so $\gamma_t(G) \leq 2|A|$. Now let $y \in A - \{x\}$ such that $|N(y) - N(x)| \geq 2$. Hence, we have:

$$n \geq 2|A| + \Delta(G)$$

$$\Rightarrow \gamma_t(G) + \Delta(G) - 1 \geq 2|A| + \Delta(G)$$

$$\Rightarrow \gamma_t(G) \geq 2|A| + 1,$$

a contradiction. Hence, for every vertex $y \in A - \{x\}$, $|N(y) - N(x)| = 1$.
Stage 3: Let $y \in A - \{x\}$ and $N(y) \cap N(x) \neq \emptyset$. Let $u \in N(y) \cap N(x)$. Now, $S = (V - N(x) \cup \{y\}) \cup \{u\}$ is a total dominating set and $|S| = n - \Delta(G)$. So, $\gamma_t(G) \leq n - \Delta(G)$, a contradiction.
Stage 4: Let $y, z \in A - \{x\}$ and $N(y) \cap N(z) \neq \emptyset$. Now $S = (V - (\{z\} \cup N(x))) \cup \{u\}$, where $u \in N(x)$, is a total dominating set and $|S| = n - \Delta(G)$. So, $\gamma_t(G) \leq n - \Delta(G)$, a contradiction. Hence, G is a graph in form of $K_{1,t} \cup rK_2$. ∎

COROLLARY 3.1 *Let T is a Tree. Then $\gamma_t(T) = n - \Delta(T) + 1$ if and only if T is a star.*

4 Total domination numbers of circular complete graphs

If n and d are positive integers with $n \geq 2d$, then circular complete graph $K_{n,d}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which v_i is adjacent to v_j if and only if $d \leq |i - j| \leq n - d$. In this section we determine the total domination of circular complete graphs. It is easy to see that $K_{n,1}$ is the complete graph K_n and $K_{n,2}$ is a circle on n vertices, therefore we assume that $d \geq 3$.

Theorem 4.5 *For $n \geq 4d - 2$ and $d \geq 3$, $\gamma_t(K_{n,d}) = 2$.*

Proof: Clearly, $\gamma_t(K_{n,d}) \geq 2$. Let $S = \{v_0, v_{2d-1}\}$. We will show that S is a total dominating set for $K_{n,d}$. Since $n \geq 4d - 2$ and $2d - 1 \leq 2d$, then $2d - 1 \leq n - d$. Also $2d - 1 \geq d$ since $d \geq 3$. Thus $d \leq 2d - 1 \leq n - d$ and $v_0 v_{2d-1} \in E(K_{n,d})$. By definition of $K_{n,d}$, v_0 is adjacent to each of the vertices $v_d, v_{d+1}, \ldots, v_{n-d}$.

Now for each $1 \leq i \leq d - 1$ we have

$$n - d + i - (2d - 1) = n - 3d + i + 1 \geq 4d - 2 - 3d + i + 1 \geq d$$

and

$$n - d + i - (2d - 1) = n - 3d + i + 1 \leq n - 3d + d = n - 2d < n - d.$$

Thus v_{2d-1} is adjacent to each of the vertices $v_{n-d+1}, \ldots, v_{n-1}$. On the other hand, for each $1 \leq i \leq d-1$ we have

$$2d-1-i \leq 2d-2 \leq 3d-2 \leq n-d$$

and

$$2d-1-i \geq 2d-1-d+1 = d.$$

Hence v_{2d-1} is adjacent to each of the vertices $v_0, v_1, \ldots, v_{d-1}$ and so S is a total dominating set for $K_{n,d}$ and $\gamma_t(K_{n,d}) = 2$. ■

Theorem 4.6 *For $3d \leq n \leq 4d-3$ and $d \geq 3$, $\gamma_t(K_{n,d}) = 3$.*

Proof: Let $S = \{v_0, v_d, v_{2d-1}\}$. We prove that S is a $\gamma_t(K_{n,d})$- set. Since $d \leq 2d-2 \leq n-d$, $G[S]$ contains no isolated vertices. Clearly v_0 and v_d are adjacent to each of the vertices $v_d, v_{d+1}, \ldots, v_{n-d}$ and $v_{2d}, v_{2d+1}, \ldots, v_{n-d}$ respectively. For $1 \leq i \leq d-1$ we have

$$2d-1-i \leq 2d-1-d+1 = d$$

and

$$2d-1-i \leq 2d-2 \leq 2d \leq n-d$$

Thus v_{2d-1} is adjacent to each of the vertices $v_1, v_2, \ldots, v_{d-1}$. Hence S is a total dominating set for $K_{n,d}$ and so $\gamma_t(K_{n,d}) \leq 3$. Now we prove that there is no total dominating set for $K_{n,d}$ of size 2. Let $S' = \{u, v\}$ be a $\gamma_t(K_{n,d})$-set. Without loss of generality, let $u = v_0$ and $v = v_j$. Clearly $d \leq j \leq n-d$. Since $v_0 v_{n-d+1} \notin E(K_{n,d})$, $d \leq n-d+1-j \leq n-d$ and so $1 \leq j \leq d+1$. Thus $j = d$ or $j = d+1$. In both cases, S' is not a total dominating set since $v_2, v_3, \ldots, v_{d-1}$ are not dominated by S' a contradiction. This completes the proof. ■

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