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A Numerical Method for Solving Linear Integral Equations

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Abstract

Integral equations find special applicability within scientific and mathematical disciplines. A powerful and efficient homotopy methodology in solving linear integral equations is presented. To check the numerical method, it is applied to solve different test problems with known exact solutions and the numerical solutions obtained confirm the validity of the numerical method and suggest that it is an interesting and viable alternative to existing numerical methods for solving the problem under consideration. Convergence is also observed

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1 Introduction

The homotopy analysis method (HAM) was proposed by Liao [9–12]. In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping, and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. The approximations obtained by the HAM are uniformly valid not only for small parameters, but also for very large parameters. Until recently, the application of the homotopy analysis method in nonlinear problems has been devoted by scientists and engineers.

Some different valid methods for solving integral equation have been developed in the last years [1–8].

In this paper, we present an iterative scheme based on the HAM for the first and second kind of linear Fredholm and Volterra integral equations

$$\begin{aligned} g(x) &= \lambda \int_a K(x,t)y(t)dt, \\ y(x) &= g(x) + \lambda \int_a K(x,t)y(t)dt, \end{aligned}$$

where the upper limit may be either variable or fixed, λ is a complex number, the kernel $K(x,t)$ and $g(x)$ are known functions, whereas y is to be determined.

In the Banach space $C[a,b]$, the Fredholm integral operator B defined by $By = \int_a^b K(x,t)y(t)dt$ is a bounded linear operator. The iterated integral operators $B^m y$ with iterated kernel $K_m(x,t)$ are defined as follows

$$\begin{aligned} B^0 y &= Iy = y, \\ By &= \int_a^b K(x,t)y(t)dt, \\ B^m y &= \int_a^b K_m(x,t)y(t)dt, \qquad m = 1,2,3,\dots, \end{aligned} \tag{1}$$

where $K_1(x,t) = K(x,t)$ and for $m = 2,3,\dots$

$$K_m(x,t) = \int_a^b K_{m-1}(x,\tau)K(\tau,t)d\tau = \int_a^b K(x,\tau)K_{m-1}(\tau,t)d\tau.$$

Voltera integral operator is defined as the same as Fredholm integral operator by replacing b by x and the Voltera iterated kernel is

$$K_m(x,t) = \int_t^x K_{m-1}(x,\tau)K(\tau,t)d\tau = \int_t^x K(x,\tau)K_{m-1}(\tau,t)d\tau.$$

Some examples are tested, and the obtained results suggest that newly improvement technique introduces a promising tool and powerful improvement for solving integral equations.

2 Description of the Method

Consider

$$N[y] = 0,$$

where N is an operator, $y(x)$ is unknown function and x the independent variable. Let $y_0(x)$ denote an initial guess of the exact solution $y(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct such a homotopy

$$(1-q)L[\phi(x;q) - y_0(x)] - qhH(x)N[\phi(x;q)] = \hat{H}[\phi(x;q); y_0(x), H(x), h, q]. \tag{2}$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.

Enforcing the homotopy (2) to be zero, i.e.,

$$\hat{H}[\phi(x;q); y_0(x), H(x), h, q] = 0$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - y_0(x)] = qhH(x)N[\phi(x;q)]. \tag{3}$$

When $q = 0$, the zero-order deformation equation (3) becomes

$$\phi(x;0) = y_0(x), \tag{4}$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the zero-order deformation equation (3) is equivalent to

$$\phi(x;1) = y(x). \tag{5}$$

Thus, according to (4) and (5), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such a kind of continuous variation is called deformation in homotopy.

By Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m \tag{6}$$

where

$$y_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;q)}{\partial q^m}|_{q=0}. \tag{7}$$

If the initial guess $y_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h, and the auxiliary function $H(x)$ are properly chosen so that the power series (6) of $\phi(x;q)$ converges at $q = 1$.Then, we have under these assumptions the solution series

$$y_m(x) = \phi(x;1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{8}$$

For brevity, define the vector

$$\overrightarrow{y}_n(x) = \{y_0(x), y_1(x), y_2(x), \ldots, y_n(x)\}. \tag{9}$$

According to the definition (7), the governing equation of $y_m(x)$ can be derived from the zero-order deformation equation (3). Differentiating the zero-order deformation equation (3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called mth-order deformation equation

$$\begin{aligned} L[y_m(x) - \chi_m y_{m-1}(x)] &= hH(x)\Re_m(\overrightarrow{y}_{m-1}(x)), \\ y_m(0) &= 0, \end{aligned} \tag{10}$$

where

$$\Re_m(\overrightarrow{y}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0} \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

Note that the high-order deformation equation (10) is governing by the linear operator L, and the term $R_m(\overrightarrow{y}_{m-1}(x))$ can be expressed simply by (11) for any nonlinear operator N.

Therefore, $y_m(x)$ can be easily gained, especially by means of computational software such as MATLAB. The solution $y(x)$ given by the above approach is dependent of L, h, $H(x)$, and $y_0(x)$. Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined. If $\sum\limits_{m=0}^{n} y_m(x)$ tends uniformly to a limit as $n \to \infty$, then this limit is the required solution.

3 Linear Integral Equations of the First Kind

Consider the linear integral equation

$$g(x) = \lambda \int_a K(x,t)y(t)dt, \tag{12}$$

where the upper limit may be either variable or fixed, λ is a complex number, the kernel $K(x,t)$ and $g(x)$ are known functions, whereas y is to be determined. Let

$$N[y] = g(x) - \lambda \int_a K(x,t)y(t)dt = 0,$$

we can obtain from (11) that

$$\Re_m(\overrightarrow{y}_{m-1}(x)) = (1-\chi_m)g(x) - \lambda \int_a K(x,t)y_{m-1}(t)dt.$$

The mth-order deformation equation (10) reduces to

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)[(1-\chi_m)g(x) - \lambda \int_a K(x,t)y_{m-1}(t)dt]. \tag{13}$$

Choose $Ly = y$ as the auxiliary linear operator, as a zero-order approximation to the desired function $y(x)$, the solution $y_0(x) = g(x)$, is taken, the nonzero auxiliary parameter h and the auxiliary function $H(x)$, can be taken as $h = 1$ and $H(x) = 1$. This is substituted into (13) to obtain the following simple iteration formula for $y_m(x)$

$$\begin{aligned} y_0(x) &= g(x) \\ y_m(x) &= y_{m-1}(x) - \lambda \int_a K(x,t)y_{m-1}(t)dt, \qquad m = 1, 2, \ldots . \end{aligned} \tag{14}$$

By considering the notations in (1), we can verify that

$$\begin{aligned} y_1(x) &= g - \lambda Bg = (I - \lambda B)g, \\ y_2(x) &= y_1 - \lambda B y_1 = (I - \lambda B)y_1 = (I - \lambda B)^2 g, \\ &\vdots \\ y_m(x) &= y_{m-1} - \lambda B y_{m-1} = (I - \lambda B)^m g. \end{aligned}$$

The solution $y(x)$ of (12), becomes

$$\begin{aligned} y(x) &= g(x) + \sum_{m=1}^{\infty} y_m(x) \qquad (15) \\ &= g(x) + \sum_{m=1}^{\infty} (I - \lambda B)^m g \\ &= g(x) + \sum_{m=1}^{\infty} \left(g(x) + \sum_{j=1}^{m} \binom{m}{j} (-\lambda)^j \int_a K_j(x,t) g(t) dt \right). \end{aligned}$$

If $\|I - \lambda B\| < 1$, then the uniform convergence of the series (15) is ensured.

Theorem 1 *As long as the series (8) convergence, where $y_m(x)$ is governed by Eq.(13), it must be the exact solution of the integral equation (12).*

Proof. If the series (8) converges, we can write

$$S(x) = \sum_{m=0}^{\infty} y_m(x),$$

and it holds that

$$\lim_{m \to \infty} y_m(x) = 0.$$

We can verify that

$$\begin{aligned} \sum_{m=1}^{n} [y_m(x) - \chi_m y_{m-1}(x)] &= y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) \qquad (16) \\ &= y_n(x), \end{aligned}$$

which gives us, according to (16),

$$\sum_{m=1}^{\infty} [y_m(x) - \chi_m y_{m-1}(x)] = \lim_{n \to \infty} y_n(x) = 0. \qquad (17)$$

Furthermore, using (17) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = L[\sum_{m=1}^{\infty} [y_m(x) - \chi_m y_{m-1}(x)]] = 0.$$

In this line, we can obtain that

$$\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x) \sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(x)) = 0$$

which gives, since $h \neq 0$ and $H(x) \neq 0$, that

$$\sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(x)) = 0. \tag{18}$$

Substituting $\Re_{m-1}(\overrightarrow{y}_{m-1}(x))$ into the above expression and simplifying it, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(x)) &= \sum_{m=1}^{\infty}[(1-\chi_m)g - \lambda \int_a^b K(x,t) y_{m-1}(t)dt] \qquad (19)\\
&= g(x) - \lambda \int_a^b K(x,t)[\sum_{m=1}^{\infty} y_{m-1}(t)]dt \\
&= g(x) - \lambda \int_a^b K(x,t)[\sum_{m=0}^{\infty} y_m(t)]dt \\
&= g(x) - \lambda \int_a^b K(x,t) S(t)dt
\end{aligned}$$

From (19) and (18), we have

$$g(x) = \lambda \int_a^b K(x,t) S(t)dt,$$

and so, $S(x)$ must be the exact solution of Eq. (12). ■

4 Linear Integral Equations of the Second Kind

Consider the linear integral equation

$$y(x) = g(x) + \lambda \int_a K(x,t) y(t)dt, \tag{20}$$

where the upper limit may be either variable or fixed, λ is a complex number, the kernel $K(x,t)$ and $g(x)$ are known functions, whereas y is to be determined. Let

$$N[y] = y(x) - g(x) - \lambda \int_a K(x,t)y(t)dt = 0,$$

we can get from (11) that

$$\Re_m(\overrightarrow{y}_{m-1}(x)) = y_{m-1}(x) - \lambda \int_a K(x,t)y_{m-1}(t)dt - (1-\chi_m)g(x).$$

In this line the mth-order deformation equation (10) has the form

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)[y_{m-1}(x) - \lambda \int_a K(x,t)y_{m-1}(t)dt - (1-\chi_m)g(x)]. \tag{21}$$

Next, we will show that the well-known method of successive approximations used to solve Eq. (20), can be obtained by the HAM approach. For this purpose, we take an initial guess $y_0(x) = g(x)$, an auxiliary linear operator $Ly = y$, a nonzero auxiliary parameter $h = -1$, and an auxiliary function $H(x) = 1$. This is substituted into (21) to give the recurrence relation

$$\begin{aligned} y_0(x) &= g(x) \qquad (22) \\ y_m(x) &= \lambda \int_a K(x,t)y_{m-1}(t)dt. \end{aligned}$$

By considering the notations in (1), we have from (22) that

$$\begin{aligned} y_1(x) &= \lambda B y_0 = \lambda B g, \\ y_2(x) &= \lambda B y_1 = (\lambda B)^2 g, \\ &\vdots \\ y_m(x) &= \lambda B y_{m-1} = (\lambda B)^m g. \end{aligned}$$

The solution $y(x)$ becomes

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} y_m(x) \qquad (23) \\ &= \sum_{m=0}^{\infty} (\lambda B)^m g \\ &= g(x) + \lambda \int_a K_1(x,t) g(t) dt + \lambda^2 \int_a K_2(x,t) g(t) dt + \cdots \\ &= g(x) + \sum_{m=1}^{\infty} \lambda^m \int_a K_m(x,t) g(t) dt, \end{aligned}$$

which is the method of successive approximations. If $|\lambda| \, \|B\| < 1$, then the series solution (23) convergence uniformly

Theorem 2 *As long as the series (8) convergence, where $y_m(x)$ is governed by Eq.(21), it must be the exact solution of the integral equation (20).*

Proof. The proof is similar to that of Theorem 1. ■

5 Numerical Results and Discussion

The HAM provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution. The consequent series truncation, and the practical procedure conducted to accomplish this task, together transforms the analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In order to investigate the accuracy of the HAM solution with a finite number of terms, three examples were solved.

To show the efficiency of the present method for our problem in comparison with the exact solution we report absolute error which is defined by

$$|Ey_{HAM}^m| = |y_{exact} - y_{HAM}^m|$$

where $y_{HAM}^m = \sum_{i=0}^{m} y_i(x)$. MATLAB 7 is used to carry out the computations.

Example 1. *Consider the Voltera integral equation of the second kind*

$$y(x) = (1+x) + \int_0^x (x-t) \; y(t) \; dt.$$

For which the exact solution is $y(x) = e^x$. *We begin with* $y_0(x) = 1 + x$. *Its iteration formulation reads*

$$y_m(x) = \int_0^x (x - t)\ y_{m-1}(t)dt, \qquad m = 1, 2, \ldots .$$

Some numerical results of these solutions are presented in Table 1.

Table 1.
Numerical results of Example 1

x_i	y_{exact}	y_{HAM}^{10}	$\lvert y_{exact} - y_{HAM}^{10}\rvert$
0	1	1	0
0.1	1.10517091807565	1.10517091807565	$4.4408920985E-16$
0.2	1.22140275816017	1.22140275816017	$4.4408920985E-16$
0.3	1.34985880757600	1.34985880757600	0
0.4	1.49182469764127	1.49182469764127	$2.2204460492E-16$
0.5	1.64872127070013	1.64872127070013	0
0.6	1.82211880039051	1.82211880039051	$2.2204460492E-16$
0.7	2.01375270747048	2.01375270747048	0
0.8	2.22554092849247	2.22554092849247	0
0.9	2.45960311115695	2.45960311115695	$4.4408920985E-16$
1	2.71828182845905	2.71828182845905	$4.4408920985E-16$

Example 2. *Consider the linear Fredholm integral equation of the second kind*

$$y(x) = \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\ y(t)\ dt$$

Beginning with $y_0(x) = \cos x$. *Its iteration formula reads*

$$y_m(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\ y_{m-1}(t)\ dt, \qquad m = 1, 2, \ldots .$$

Ten, fifteen and twenty terms approximation of $y(x)$ *are*

$$\sum_{m=0}^{10} y_m(x) = \cos x + 0.998 \sin x$$

$$\sum_{m=0}^{15} y_m(x) = \cos x + 0.9999 \sin x$$

$$\sum_{m=0}^{20} y_m(x) = \cos x + \sin x$$

The exact solution $y(x) = \cos x + \sin x$ is obtained.

Example 3. *Consider the Fredholm integral equation of the first kind*

$$\frac{1}{2}(e-1)e^x = \int_0^{\frac{1}{2}} e^{x+t}\, y(t)\, dt.$$

For which the exact solution is $y(x) = e^x$. We begin with $y_0(x) = \frac{1}{2}(e-1)e^x$. Its iteration formulation reads

$$y_m(x) = y_{m-1} - \int_0^{\frac{1}{2}} e^{x+t}\, y_{m-1}(t)dt, \qquad m = 1, 2, \ldots .$$

Some numerical results of these solutions are presented in Table 2.

Table2.
Numerical results of Example 3

x_i	y_{exact}	y^{15}_{HAM}	$\lvert y_{exact} - y^{15}_{HAM}\rvert$
0	1.000000000000	0.999999996924	$3.07571745850E-9$
0.1	1.105170918075	1.105170914677	$3.39855521502E-9$
0.2	1.221402758160	1.221402754403	$3.75716813216E-9$
0.3	1.349858807576	1.349858803425	$4.15055501079E-9$
0.4	1.491824697641	1.491824693052	$4.58839011230E-9$
0.5	1.648721270700	1.648721265629	$5.07087261070E-9$

6 Conclusion

The proposed method is a powerful procedure for solving linear integral equations. The examples analyzed illustrate the ability and reliability of the method presented in this paper and reveals that this one is very simple and effective. The obtained solutions, in comparison with exact solutions admit a remarkable accuracy. Results indicate that the convergence rate is very fast, and lower approximations can achieve high accuracy.

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