

Covering and Metric Regularity

Alexandru V. Blaga

National College "Mihai Eminescu"
Mihai Eminescu Street, Satu Mare 440014, Romania
alblaga2005@yahoo.com

Abstract. We consider some theorems on the covering property of mappings defined between Banach spaces. Also we study some related properties and propose nonlocal theorems of the covering-type or contract. It is known that the covering property and metric regularity property are equivalent. The author's opinion is that among these two properties, the more convenient for applications, for usage in concrete situations is metric regularity (the distance estimate to the level sets), whereas the more convenient to prove is covering.

Mathematics Subject Classification: Primary: 49J52, 49J53. Secondary: 46G05, 49J50, 90C31

Keywords: Covering and metric regularity, contract and metric regularity, strictly differentiable, the covering rate, the contract rate, local and nonlocal covering, nonlinear operators, Lyusternik Theorem

1. INTRODUCTION

The metric regularity is a notion which was introduced in the years 1930–1960 and it has developed a lot during the last years. The famous theorem of Lyusternik–Graves [2] founded the notion of metric regularity and the covering with linear rate of set. Under the condition in which the function $F : X \rightarrow Y$, $x_0 \in X$, X and Y Banach spaces is strictly differentiable at x_0 and $F'(x_0)$ is a surjective operator, the property of metric regularity on a neighborhood is assured from Theorem 1.1 [2]. If $y_0 = F(x_0)$, in the conditions of Theorem 1.1 [2], we have the relation:

$$(1.1.1) \quad d(x, F^{-1}(y)) \leq L \cdot \|F(x) - y\|,$$

for all $x \in U$ with $U \in \mathcal{V}(x_0)$ and $y \in V$, with $V \in \mathcal{V}(y_0)$.

We have for a set $A \subset X$, $d(x, A) = \inf_{y \in A} d(x, y)$.

The property expressed by the relation (1.1) is in fact the metric regularity with a L constant, which is equivalent with the next property:

There is $a > 0$ and a neighborhood of x_0 so that for each closed ball $B_r(x) \subset U$, we have:

$$(1.1.2) \quad F(B_r(x)) \supset B_{ar}(F(x)).$$

The relation expressed the property of covering with linear rate a of the set U . The equivalence of the two properties for an arbitrary and continuous function F means:

- (i) if F covers $U \in \mathcal{V}(x_0)$ with the rate $a > 0$, then for a random neighborhood $V_1 \in \mathcal{V}(x_0)$ and $V_2 \in \mathcal{V}(y_0)$ we have the relation (1.1) for $L = \frac{1}{a}$.
- (ii) if F is metrically regular with the constant L , for $V_1 \in \mathcal{V}(x_0)$ and $V_2 \in \mathcal{V}(y_0)$ then for a random neighborhood $U \in \mathcal{V}(x_0)$, F covers U with the rate $a < \frac{1}{L}$.

If $F : \mathbb{R} \rightarrow \mathbb{R}$, the both properties of covering and metric regularity mean the following: on the neighborhood $U \in \mathcal{V}(x_0)$, we have $F'(x) \leq a$, $F'(x) \leq \frac{1}{L}$ and $a = \frac{1}{L}$ for the area x belonging to U . The aim of this article is to present some results of the covering in linear rates which imply the metric regularity. We denote the metrics in X and Y by the same letter d and the ball $B_r(x)$ is sometimes denoted by $B(x, r)$.

Definition 1.1. *If X is a complete metric space, Y a normed space and $T : X \rightarrow Y$, we say that mapping T covers the set $G \subset X$ with the rate $a > 0$ if*

$$(1.1.3) \quad T(B_r(x)) \supset B_{ar}(T(x)),$$

for all balls $B_r(x) \subset G$.

Definition 1.2. *The application $S : X \rightarrow Y$ where X and Y are given by the Definition 1.1 contracts on $G \subset X$ with the rate $b > 0$ if*

$$(1.1.4) \quad S(B_r(x)) \subset B_{br}(S(x)),$$

for all balls $B_r(x) \subset G$.

It is obvious that if S contracts on G , S is continuous on G and mutual any application with properties b -Lipschitzian on G , contracts on G with the rate b .

2. MAIN RESULTS

Next, we state two important theorems which will be used in the operations with operators that cover and contract on the set G .

Theorem 2.1. ([2]) *Let the mapping $F : X \rightarrow Y$, where the spaces X and Y are given by the Definition 1.1 and $x_0 \in X$ so that:*

- (i) F is strictly differentiable on x_0
- (ii) $F'(x_0)$ is an surjective operator.

Then there exists a constant L and neighborhoods $U \in \mathcal{V}(x_0)$ and $V \in \mathcal{V}(y_0)$ in order to have the relation (1.1).

Theorem 2.2. (Milyutin, Dmitruk, Osmolovskii, [2]) *Let T be continuous on G and covers G with the rate $a > 0$ and let S contracts on G with the rate $b < a$. Then their sum $F = T + S$ covers on G with the rate $a - b > 0$. The assumption of continuity of T can be weakened to the closedness of its graph on G . In this case the graph of F on G is also closed.*

Theorem 2.3. *Let X be a Banach space, Y a normed space and $G \subset X$ which is covered by the mapping $T : X \rightarrow Y$ with the rate $a > 0$. If F is linear, continuous and bijective, then*

$$(2.2.1) \quad F(\overline{B_r(x)}) \supset \overline{F(B_r(x))} \supset \overline{B_{ar}(F(x))},$$

for all $B_r(x) \subset G$.

Proof. Let $y \in \overline{B_{ar}(F(x))}$, so there exists $y_n \in B_{ar}(F(x))$ with the property that $\lim_{n \rightarrow \infty} y_n = y$. So $B_{ar}(F(x)) \subset F(B_r(x))$ is a relation which takes place from the property of the covering of F with the rate a . From this we deduce the existence of a sequence $x_n \in B_r(x)$, with $F(y_n) = x_n$, $\|x_n - x\| \leq r$, $\|y_n - F(x)\| \leq ar$. From $F(y_n) = x_n$, we have $y_n = F^{-1}(x_n)$ and from the continuity of F which is inversable and linear we deduce the continuity of F^{-1} . So we have $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} F^{-1}(y_n) = F^{-1}(y) = \hat{x}$, $\hat{x} \in \overline{B_r(x)}$. From $y = F(\hat{x})$ with $\hat{x} \in \overline{B_r(x)}$, we have the relation:

$$\overline{B_{ar}(F(x))} \subset \overline{F(B_r(x))}.$$

On the other hand, we have $B_{a \cdot r}(F(x)) \subset F(B_r(x))$ from the covering property of f . We deduce that $\overline{B_{a \cdot r}(F(x))} \subset \overline{F(B_r(x))}$. Let $u \in \overline{F(B_r(x))}$ be so there is $u_n \in F(B_r(x))$, with $\lim_{n \rightarrow \infty} u_n = u$ and $u_n = F(z_n)$, $z_n \in B_r(x) \subset \overline{B_r(x)}$. But $z_n = F^{-1}(u_n)$ and z_n is convergent, so there is $\lim_{n \rightarrow \infty} F^{-1}(u_n)$. If $z = \lim_{n \rightarrow \infty} z_n$, we have $F^{-1}(u) = z$ that is $u = F(z)$ or $u \in \overline{F(B_r(x))}$, because $z \in \overline{B_r(x)}$. \square

Inclusions from relation (2.1) take place for an optimal radius r given by this ball, but r cannot be replaced with $r_1 > r$ because we have:

Theorem 2.4. *Let X be a Banach space, Y a normed space and the mapping $F : X \rightarrow Y$ which covers the set $G \subset X$ with the rate $a > 0$. If F is linear, continuous and bijective, we have*

$$(2.2.2) \quad F(B_{r_1}(x) \setminus \overline{B_r(x)}) \cap \overline{B_{ar}(F(x))} = \emptyset.$$

Proof. Let $y \in \overline{B_{ar}(F(x))}$, so there is $y_n \in B_{ar}(F(x))$ for which $\lim_{n \rightarrow \infty} y_n = y$. Because F covers G with the rate $a > 0$, from the relation $B_{ar}(F(x)) \subset F(B_r(x))$, we deduce the existence of a sequence $x_n \in B_r(x)$ and $F(x_n) = y_n$. Because the sequence $\{y_n\}_{n \geq 0}$ is convergent and F is continuous and inversable so with its inverse continuous, we deduce that the sequence $\{x_n\}_{n \geq 0}$ is convergent. We have $\lim_{n \rightarrow \infty} x_n = \hat{x}$, $y = F(\hat{x})$, $\hat{x} \in \overline{B_r(x)}$. If $y \in F(B_{r_1}(x) \setminus \overline{B_r(x)})$, so there is $x_1 \in B_{r_1}(x)$ and $x_1 \notin \overline{B_r(x)}$, with $F(x_1) = y$. Since F is injective, we deduce $x_1 = \hat{x}$, contradiction. \square

Theorem 2.5. *Let X, Y, Z be Banach spaces and the mappings $F_1 : X \rightarrow Y$, $F_2 : Y \rightarrow Z$ so that:*

- (i) F_1 covers $G_1 \subset X$ with the rate $a_1 \geq 1$
- (ii) F_2 covers $G_2 = F_1(G_1) \subset Y$ with the rate $a_2 > 0$.

Then the mapping $F_2 \circ F_1 : X \rightarrow Z$ covers G_1 with the rate a_2 .

Proof. From the property of covering F_1 , we deduce that $B_r(F_1(y)) = B_{a_1 \cdot \frac{r}{a_1}}(F_1(y)) \subset F_1\left(B_{\frac{r}{a_1}}(y)\right)$, because $B_{\frac{r}{a_1}}(y) \subset B_r(y)$, from the condition $a_1 \geq 1$. From the property of covering F_2 , we have: $B_{a_2 \cdot r}(F_2(F_1(y))) \subset F_2(B_r(F_1(y))) \subset (F_2 \circ F_1)\left(B_{\frac{r}{a_1}}(y)\right) \subset (F_2 \circ F_1)(B_r(y))$. This sequence of inclusions is well defined because $a_1 \geq 1$. \square

Theorem 2.6. *Let X, Y, Z be Banach spaces and the mappings $S_1 : X \rightarrow Y$, $S_2 : Y \rightarrow Z$ so that:*

- (i) S_1 contracts $G_1 \subset X$ with the rate $b_1 \geq 0$
- (ii) S_2 contracts $G_2 = S_1(G_1) \subset Y$ with the rate $0 < b_2 \leq 1$.

Then the mapping $S_2 \circ S_1 : X \rightarrow Z$ contracts on G_1 with the rate b_2 .

Proof. We want to prove that $B_{b_2 r}((S_2 \circ S_1)(y)) \supset (S_2 \circ S_1)(B_r(y))$, for any ball which verifies $B_r(y) \subset G_1$. We have $B_{b_2 r}(S_2(S_1(y))) \supset S_2(B_r(S_1(y)))$, because S_2 contracts on G_2 with the rate b_2 . But on the other hand, we have $B_r(S_1(y)) = B_{\frac{r}{b_1} \cdot b_1}(S_1(y)) \supset S_1\left(B_{\frac{r}{b_1}}(y)\right) \supset S_1(B_r(y))$, because $b_1 \leq 1$ and

S_1 contracts on G_1 with the rate b_1 . Then we can write $B_{br}((S_2 \circ S_1)(y)) \supset S_2(B_r(S_1(y))) \supset (S_2 \circ S_1)(B_r(y))$. \square

Remark 2.1. *If $F : X \rightarrow Y$ is a bijective and $G \subset X$ so that $F(G) = G$ and F covers G with the rate $a > 0$, then F contracts on G with the rate $1/a$.*

Theorem 2.7. *Let X, Y be Banach spaces and the mappings $S_1 : X \rightarrow Y$, $S_2 : X \rightarrow Y$ and $G \subset X$ so that:*

- (i) S_1 contracts on G with the rate b_1
- (ii) S_2 contracts on G with the rate b_2 .

Then the mapping $S_1 + S_2 = T : X \rightarrow Y$ contracts on G with the rate $b_1 + b_2$.

Proof. If $S_1 + S_2 = T$ and $b_1 + b_2 = b$ we have to prove that $T(B_r(x)) \subset B_{br}(T(x))$, for any $B_r(x) \subset G$. Let $y \in T(B_r(x))$ so $y = T(u)$ with $u \in B_r(x)$, $\|u - x\| \leq r$. We have $\|T(u) - T(x)\| = \|(S_1 + S_2)(u) - (S_1 + S_2)(x)\| = \|S_1(u) - S_1(x) + S_2(u) - S_2(x)\| = \|S_1(u) - S_1(x)\| + \|S_2(u) - S_2(x)\| \leq (b_1 + b_2)r = br$. We have taken into account the contract property of S_1 and S_2 , that is $S_1(B_r(x)) \subset B_{b_1 r}(S_1(x))$, $S_2(B_r(x)) \subset B_{b_2 r}(S_2(x))$, from where we deduce that $\|S_1(u) - S_1(x)\| \leq b_1 r$, $\|S_2(u) - S_2(x)\| \leq b_2 r$. \square

Theorem 2.8. *Let X, Y be Banach spaces and the mappings $T : X \rightarrow Y$, $S : X \rightarrow Y$ and $\alpha \in (0, 1)$ so that:*

- (i) T covers on G with the rate $a > 0$
- (ii) T is continuous on G
- (iii) S contracts on G with the rate b
- (iv) $\frac{b}{a+b} < \alpha < 1$.

Then the mapping $F = \alpha T + (1 - \alpha)S : X \rightarrow Y$ covers on G with the rate $\alpha a - (1 - \alpha)b$.

Proof. We will prove that the mapping $\alpha \cdot T$ covers on G with the rate $\alpha \cdot a$ and the mapping $(1 - \alpha)S$ contracts on G with the rate $(1 - \alpha)b$. Let $y \in B_{\alpha \cdot a \cdot r}(\alpha T(x))$ be, that is $\|y - \alpha T(x)\| \leq \alpha \cdot a \cdot r$, which is equivalent with $\frac{y}{\alpha} - T(x) \leq ar$, so $\frac{y}{\alpha} \in B_{ar}(T(x)) \subset T(B_r(x))$, because T covers on G . From $\frac{y}{\alpha} \in T(B_r(x))$ we deduce that $\frac{y}{\alpha} = T(u)$ with $u \in B_r(x)$, or $y = \alpha T(u) \in \alpha T(B_r(x))$, so $B_{\alpha \cdot a \cdot r}(\alpha T(x)) \subset \alpha T(B_r(x))$. If $z \in (1 - \alpha)S(B_r(x))$, we have $z = (1 - \alpha)S(u)$, with $u \in B_r(x)$ and $\|z - (1 - \alpha)S(x)\| = \|S(x) - S(u)\|(1 - \alpha) \leq (1 - \alpha)br$ and this because $S(B_r(x)) \subset B_{b \cdot r}(S(x))$. We deduce that $z \in B_{(1-\alpha) \cdot b \cdot r}((1 - \alpha)S(x))$. From Theorem 2.2 we deduce that the mapping F covers on G with the rate $\alpha a - (1 - \alpha)b > 0$, because $\alpha > \frac{b}{a+b}$. \square

Theorem 2.9. *Let W, Y, Z be Banach spaces and the mappings $T : W \rightarrow Y$, $S : W \rightarrow Z$ so that:*

- (i) T is linear and contracts on $G \subset W$ with the rate $a \geq 0$
- (ii) S is linear and contracts on G with the rate $b > 0$.

Then the mapping $F : W \rightarrow Y \times Z$ defined by $F(w) = (T(w), S(w))$ contracts on G with the rate $\min\{a + b, \|T\| + \|S\|\}$.

Proof. If $\theta \in F(B_r(x_0))$, $B_r(x_0) \subset G$, we have $\theta = F(u)$, where $u \in B_r(x_0)$. We have $\|F(u) - F(x_0)\| = \|(T(u) - T(x_0), S(u) - S(x_0))\| = \|T(u) - T(x_0)\| + \|S(u) - S(x_0)\| \leq \|T\| \cdot \|u - x_0\| + \|S\| \cdot \|u - x_0\| \leq (\|T\| + \|S\|)r$. We have proven that $F(u) \in B_{(\|T\| + \|S\|)r}(F(x_0))$. On the other hand, from the property to contract of T and S on G we have $\|T(u) - T(x_0)\| \leq ar$, because $T(B_r(x_0)) \subset B_{ar}(T(x_0))$ and $\|S(u) - S(x_0)\| \leq br$, because $S(B_r(x_0)) \subset B_{br}(S(x_0))$. So $\|F(u) - F(x_0)\| \leq r(a + b)$, that is $F(u) \in B_{r(a+b)}(F(x_0))$. \square

In the property on G to contract, we are interested the ball to have the radius as small as possible, so we choose the smaller of the two.

Remark 2.2. *A more complete property is deduced in Lemma 4.1 from [2], but for operators with cover property.*

Theorem 2.10. *Let X, Y, Z be Banach spaces, $G_1 \subset X$, $G_2 \subset Y$ and the mappings $T : X \rightarrow Z$, $S : Y \rightarrow Z$ so that:*

- (i) T covers on G_1 with the rate a
- (ii) S contracts G_2 with the rate $b < a$
- (iii) T is continuous on G .

Then the mapping $D : X \times Y \rightarrow Z$ defined by the relation

$$(2.2.3) \quad D((x, y)) = T(x) + S(y)$$

covers on $G_1 \times G_2$ with the rate $a - b$.

Proof. We define the mappings given by the relations:

$$(2.2.4) \quad \begin{aligned} f : X \times Y &\rightarrow X, & f((x, y)) &= x, & \forall (x, y) \in X \times Y \\ h : X \times Y &\rightarrow Y, & h((x, y)) &= y, & \forall (x, y) \in X \times Y \end{aligned}$$

and we have $(T \circ f + S \circ h)(x, y) = T(f(x, y)) + S(h(x, y)) = T(x) + S(y) = D(x, y)$. We will prove that $T \circ f$ covers on $G_1 \times G_2$ with the rate a and $S \circ h$ contracts on $G_1 \times G_2$ with the rate b , so we have

$$(2.2.5) \quad \begin{aligned} (S \circ h)(B_r(w_0)) &\subset B_{br}((S \circ h)(w_0)), & \forall B_r(w_0) \subset G_1 \times G_2 \\ (T \circ f)(B_r(w_0)) &\supset B_{ar}((T \circ f)(w_0)), & \forall B_r(w_0) \subset G_1 \times G_2, \end{aligned}$$

where $w_0 = (x_0, y_0) \in G_1 \times G_2$.

In order to prove the relations, we need

$$(2.2.6) \quad \begin{aligned} h(B_r(w_0)) &= B_r(y_0) \\ f(B_r(w_0)) &= B_r(x_0). \end{aligned}$$

Let $t \in f(B_r(w_0))$ be, that is $t = f(u)$, $u \in B_r(w_0)$, $u = (u_1, u_2)$ for which we have $\|u - w_0\| = \|(u_1 - x_0, u_2 - y_0)\| = \|u_1 - x_0\| + \|u_2 - y_0\| \leq r$. So we deduce that $\|u_1 - x_0\| \leq r$, that is $f(u) = u_1 \in B_r(x_0)$ and $f(B_r(w_0)) \subset B_r(x_0)$. If $u_1 \in B_r(w_0)$ then for $\theta = (u_1, y_0) \in B_r(w_0)$, because: $\|\theta - w_0\| = \|u_1 - x_0\| \leq r$ and $f(\theta) = u_1$, or $u_1 \in f(B_r(w_0))$ and $f(B_r(w_0)) \supset B_r(x_0)$. We have in analogous mode $h(B_r(w_0)) = B_r(y_0)$. We have $S(h(B_r(w_0))) = S(B_r(y_0)) \subset B_{br}(S(y_0)) = B_{br}((S \circ h)(w_0))$ and this because S contracts on G_2 . From $B_r(w_0) \subset G_1 \times G_2$ we have $B_r(y_0) \subset G_2$, because from $u_2 \in B_r(y_0)$ we deduce $\|u_2 - y_0\| \leq r$ and the pair $(x_0, u_2) \in B_r(w_0) \subset G_1 \times G_2$. On the other hand, we also have the inclusions $B_{ar}((T \circ f)(w_0)) = B_{ar}(T(x_0)) \subset T(B_r(x_0)) = (T \circ f)(B_r(x_0))$, because T covers on G_1 with the rate a . From $B_r(w_0) \subset G_1 \times G_2$ we have $B_r(x_0) \subset G_1$, because from $u_1 \in B_r(w_0)$, we deduce $\|u_1 - x_0\| \leq r$ and the pair $(u_1, y_0) \in B_r(w_0) \subset G_1 \times G_2$, or $u_1 \in G_1$. The relations (2.5) have been proven. Then from the Theorem 2.2 [2] we deduce that the mapping D covers $G_1 \times G_2$ with the rate $a - b$. \square

For the special case of polynomials $P : \mathbb{R} \rightarrow \mathbb{R}$, the operator $DP(x_0)\alpha = P'(x_0) \cdot \alpha$, $\forall \alpha \in \mathbb{R}$ is surjective at the points x_0 for which $P'(x_0) \neq 0$ (see Proposition 3, [1]) and at these points P has the metric regularity property. In this case, the two properties are equivalent.

Theorem 2.11. *Let $P : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial, $G \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$ with $P'(x_0) \neq 0$. Then the mapping P covers on G with the rate $a = |P'(x_0)| > 0$.*

Proof. For $B_r(x) \subset G$, we will prove that the mapping $DP(x_0)$ verify

$$(2.2.7) \quad P'(x_0)(B_r(x)) \supset B_{a \cdot r}(P'(x_0)x),$$

where $B_r(x) = (x - r, x + r)$. Since P is a continuous mapping and $DP(x_0)$ is continuous, we have that $P'(x_0)(B_r(x))$ is an interval. We denote $P'(x_0)\theta$ the center of this interval, where $\theta \in B_r(x)$ and we have $|P'(x_0)\theta - P'(x_0)x| + a \cdot r = |P'(x_0)| \cdot |\theta - x| + |P'(x_0)| \cdot r = |P'(x_0)|(|\theta - x| + r) \leq 2|P'(x_0)|r = \sup_{u \in B_r(x)} |P'(x_0)\theta - P'(x_0)u|$. These inequalities assure the relation (2.7). \square

REFERENCES

- [1] Blaga, A. V., *The metric regularity of polynomial functions in one or several variables*, Acta Univ. Apulensis Math. Inform., **16**(2008), 71–79
- [2] Dmitruk, A. V., *On a nonlocal metric regularity of nonlinear operators*, Control and Cybernetics, **34**(3)(2005), 723–746
- [3] Dontchev, A. L., Rockafellar, R. T., Lewis, A. S., *The radius of metric regularity*, Trans. Amer. Math., **355**(2)(2002), 493–517
- [4] Dontchev, A. L., Rockafellar, R. T., *Regularity and conditioning of solution mappings in variational analysis*, Set-Valued Anal., **12**(1-2)(2007), 79–109
- [5] Ioffe, A. D., *Metric regularity and subdifferential calculus*, Uspekhi Mat. Nauk, **55**(3)(2000), 103–162
- [6] Mordukhovich, B. S., *Variational analysis and generalized differentiation*, **I,II**, Springer, Berlin, 2006

Received: September, 2008