

Upper Bounds for Partitions into k-th Powers Elementary Methods

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Abstract

Let $N_k(x) = \sum_{i=1}^{\lfloor x \rfloor} p_k(i)$, where $p_k(i)$ is the number of partitions of i into k-th powers. Therefore $p_1(n) = p(n)$ is the number of partitions of n in positive integers, $p_2(n)$ is the number of partitions of n in squares, and so. " E. M. Wright wrote three papers on the asymptotics of particular generating functions The point I wish to make in the section is that Wright's third paper on partitions into powers IS UNIQUE in the history of this subject. Its starting point and fundamental philosophy are different from anything that has come before or since" (G. E. Andrews [1]). E. M. Wright [3] obtained beau asymptotic expansions for $p_k(n)$ using analitical methods, these formulas are not simple. They have a main term (a series) and an error term $O(\exp(c \sqrt[k+1]{n}))$, where the error term is of exponentially lower order of magnitude than the main term. G. E. Andrews [1] explores the asymptotic expansions of E. M. Wright and mentions that Wright suggest that each of the main terms in his expansions has order

$$\exp(c_3 \sqrt[k+1]{n} + c_4 \log n)$$

In this article we prove using very elementary methods the simple inequality

$$N_k(x) < \exp(c_1 \log x \sqrt[k+1]{x} + c_2 \log x)$$

where c_1 and c_2 are positive constants. We also prove the simple inequalities

$$p_k(n) < \exp(c_1 \log n \sqrt[k+1]{n} + c_2 \log n) < e^{n^{\frac{1}{k+1-c}}} < e^{\sqrt[k]{n}}$$

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1 Introduction. Preliminary theorems

Let r_n be a sequence of positive numbers such that:

$$r_1 < r_2 < r_3 < \dots \quad (1)$$

$$r_n \rightarrow \infty \quad (2)$$

Let us consider the infinite linear inequality (x fixed)

$$\sum_{i=1}^{\infty} r_i x_i \leq x \quad (x \geq 0) \quad (3)$$

A solution to this inequality is a vector (x_1, x_2, \dots) , where the x_i ($i = 1, 2, 3, \dots$) are non negative integers, which satisfies the inequality.

Note that (see (2)) a vector solution has only a finite number of positive x_i and for each value of x there are a finite number of solutions.

Let $N(x)$ be the number of solutions to the inequality (3). The function $N(x)$ ($x \geq 0$) is increasing. Note that $N(0) = 1$, since in this case we have the unique solution $(0, 0, 0, \dots)$. Consequently $N(x) \geq 1$.

Let us consider the finite linear inequality (x fixed)

$$\sum_{i=1}^n r_i x_i \leq x \quad (x \geq 0) \quad (4)$$

A solution to this inequality is a vector (x_1, x_2, \dots, x_n) , where the x_i ($i = 1, 2, \dots, n$) are non negative integers, which satisfies the inequality.

Let $S_n(x)$ be the number of solutions to the inequality (4).

The following lemma is well known [2] (the proof is elementary, mathematical induction and combinatory).

Lemma 1.1 *The following asymptotic formula holds,*

$$S_n(x) \sim \frac{x^n}{n! r_1 r_2 \dots r_n} \quad (5)$$

Theorem 1.2 *We have*

$$N(x) = x^{f(x)} = e^{f(x) \log x} \quad (x > 1) \quad (6)$$

where $f(x) \rightarrow \infty$.

Proof. If $(x_1, x_2, \dots, x_{n+1})$ is a solution to the inequality

$$\sum_{i=1}^{n+1} r_i x_i \leq x$$

then $(x_1, x_2, \dots, x_{n+1}, 0, 0, \dots)$ is a solution to the inequality (3).

Hence $S_{n+1}(x) \leq N(x)$ and

$$\frac{S_{n+1}(x)}{x^n} \leq \frac{N(x)}{x^n} \tag{7}$$

Lemma 1.1 imply

$$\frac{S_{n+1}(x)}{x^n} \rightarrow \infty \tag{8}$$

(7) and (8) give

$$\frac{N(x)}{x^n} \rightarrow \infty \tag{9}$$

Clearly, if $x > 1$ we can write $N(x) = x^{f(x)}$. From (9) we obtain

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x^n} = \lim_{x \rightarrow \infty} x^{f(x)-n} = \infty \tag{10}$$

Limit (10) imply the set A of values of x such that $f(x) - n \leq 0$ is bounded, since if $x \in A$ we have $x^{f(x)-n} \leq 1$. Consequently there exists x_0 such that if $x \geq x_0$ we have $f(x) - n > 0$, that is $f(x) \rightarrow \infty$. The theorem is proved.

Theorem 1.3 *If $x \in [r_n, r_{n+1})$, $N(x) = S_n(x)$ and $N(x) \geq n$.*

Proof. If $x \in [r_n, r_{n+1})$ there exists a one-to-one correspondence between the solutions to (4) and the solutions to (3).

If (x_1, x_2, \dots, x_n) is a solution to (4) then $(x_1, x_2, \dots, x_n, 0, 0, \dots)$ is a solution to (3).

On the other hand if (x_1, x_2, x_3, \dots) is a solution to (3) then $x_{n+1} = 0, x_{n+2} = 0, \dots$ since in contrary case $\sum_{i=1}^{\infty} r_i x_i \geq r_{n+1}$.

Finally, note that (see (1)) $(0, \dots, 1, \dots, 0)$ where 1 is the i -th coordinate ($i = 1, \dots, n$) is a solution to (4). The theorem is proved.

Theorem 1.4 *If $x \in [r_n, r_{n+1})$ then*

$$N(x) \leq \left(\frac{x}{r_1} + 1\right) \dots \left(\frac{x}{r_n} + 1\right) = 1 + \sum_{\binom{n}{1}} \frac{x}{r_i} + \sum_{\binom{n}{2}} \frac{x^2}{r_i r_j} + \dots + \sum_{\binom{n}{n}} \frac{x^n}{r_1 \dots r_n}$$

Proof. Clearly, we have

$$S_n(x) \leq \left(\left[\frac{x}{r_1}\right] + 1\right) \dots \left(\left[\frac{x}{r_n}\right] + 1\right) \leq \left(\frac{x}{r_1} + 1\right) \dots \left(\frac{x}{r_n} + 1\right)$$

On the other hand, $N(x) = S_n(x)$ (theorem 1.3). The theorem is proved.

Notation. $\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!}$ denotes the number of summands in each sum and $[\alpha]$ denotes the largest integer that does not exceed α .

Let $C(x)$ be the function defined in the following way, if $x \in [r_n, r_{n+1})$ then

$$C(x) = \left(\frac{x}{r_1} + 1\right) \dots \left(\frac{x}{r_n} + 1\right) = 1 + \sum_{\binom{n}{1}} \frac{x}{r_i} + \sum_{\binom{n}{2}} \frac{x^2}{r_i r_j} + \dots + \sum_{\binom{n}{n}} \frac{x^n}{r_1 \dots r_n} \quad (11)$$

Theorem 1.4 and (11) give $N(x) \leq C(x)$. We shall call to $C(x)$ a trivial upper bound to $N(x)$. We can write

$$C(x) = x^{g(x)} \quad (12)$$

Consequently (see theorem 1.2) $f(x) \leq g(x)$ and $g(x) \rightarrow \infty$. Let $R(x)$ be a function such that if $x \geq x_0$ we have $N(x) \leq R(x)$. If we write

$$R(x) = x^{s(x)} \quad (13)$$

then $f(x) \leq s(x)$. We shall call to $R(x)$ a nontrivial upper bound to $N(x)$ if and only if $(s(x)/g(x)) \rightarrow 0$. Note that is definition imply $(R(x)/C(x)) \rightarrow 0$ since $g(x) \rightarrow \infty$.

2 Main Results

Let us consider the sequence $r_n = n^k$ where k is a positive integer. In this case, we have $N(x) = N_k(x) = \sum_{i=1}^{[x]} p_k(i)$, where $p_k(i)$ is the number of partitions of i into k -th powers.

Theorem 2.1 *There exists x_0 such that if $x \geq x_0$ the following inequality holds*

$$N_k(x) < \exp(c_1 \log x^{k+1}\sqrt{x} + c_2 \log x) \quad (14)$$

where $R(x) = \exp(c_1 \log x^{k+1}\sqrt{x} + c_2 \log x)$ is a nontrivial upper bound and,

$$c_1 = \left(1 + \frac{1}{k}\right) (k+1)^{\frac{1}{k+1}} \quad c_2 > 1 + \frac{1}{k}$$

Proof. Let us consider the inequality

$$1^k + 2^k + \dots + n'^k \geq (n+1)^k \quad (15)$$

We have that

$$1^k + 2^k + \dots + n'^k = \int_0^{n'} x^k dx + H_1(n') = \frac{1}{k+1} n'^{k+1} + H_1(n') \quad (16)$$

where $0 < H_1(n') < n'^k$. From (16) we obtain that inequality (15) holds if

$$n' = \left[(k+1)^{\frac{1}{k+1}} (n+1)^{\frac{k}{k+1}} \right] + 1 \tag{17}$$

Since $1^k < 2^k < \dots < n^k$, any subset in the set $\{1^k, 2^k, \dots, n^k\}$ with h ($h \geq n'$) elements satisfies inequality (15). Note that

$$\binom{n}{1} < \binom{n}{2} < \dots < \binom{n}{n'}$$

since $(n'/n) \rightarrow 0$. Hence if $x \in [r_n, r_{n+1}) = [n^k, (n+1)^k)$ we have (see theorem 1.3 and theorem 1.4)

$$\begin{aligned} N_k(x) &= S_n(x) < 1 + \sum_{\binom{n}{1}} \frac{x}{r_i} + \sum_{\binom{n}{2}} \frac{x^2}{r_i r_j} + \dots + \sum_{\binom{n}{n'}} \frac{x^{n'}}{r_i \dots r_m} \\ &< 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n'} x^{n'} < n' \binom{n}{n'} x^{n'} \\ &< n^{n'} x^{n'} = x^{n'(\frac{\log n}{\log x} + 1)} \leq x^{n'(\frac{1}{k} + 1)} \end{aligned}$$

That is (see (17))

$$N_k(x) < \exp \left(\log x \left(1 + \frac{1}{k} \right) \left((k+1)^{\frac{1}{k+1}} (n+1)^{\frac{k}{k+1}} + 1 \right) \right)$$

Now

$$\left(\log x (n+1)^{\frac{k}{k+1}} - \log x n^{\frac{k}{k+1}} \right) \rightarrow 0$$

Consequently

$$N_k(x) < \exp \left(\log x \left(1 + \frac{1}{k} \right) \left((k+1)^{\frac{1}{k+1}} n^{\frac{k}{k+1}} + 1 \right) + o(1) \right) \tag{18}$$

From (18) we obtain (see (13))

$$N_k(x) < \exp(c_1 \log x \sqrt[k+1]{x} + c_2 \log x) = R(x) = x^{s(x)} \tag{19}$$

That is, inequality (14). Let us consider the sequence $r_n = n^k$. If $x \in [n^k, (n+1)^k)$ we have (see (11) and (12))

$$C(x) = x^{g(x)} > \frac{x^n}{1^k 2^k \dots n^k} = x^{l(x)}$$

Therefore

$$g(x) > l(x) = n - k \frac{\log 1 + \log 2 + \dots + \log n}{\log x} \geq n - \frac{\log 1 + \log 2 + \dots + \log n}{\log n}$$

Now

$$\log 1 + \log 2 + \dots + \log n = \int_1^n \log x \, dx + H_2(n) = n \log n - n + 1 + H_2(n)$$

where $0 < H_2(n) < \log n$. Consequently

$$g(x) > n - \frac{n \log n - n + 1 + H_2(n)}{\log n} > \frac{n}{\log n} - 2 \quad (20)$$

Inequalities (19) and (20) give

$$\frac{s(x)}{g(x)} < \frac{c_1(n+1)^{\frac{k}{k+1}} + c_2}{\frac{n}{\log n} - 2}$$

That is $(s(x)/g(x)) \rightarrow 0$. Therefore the upper bound $R(x)$ is a nontrivial upper bound. The theorem is proved.

Corollary 2.2 *There exists n_0 such that if $n \geq n_0$ then*

$$p_k(n) < \exp(c_1 \log n \sqrt[k+1]{n} + c_2 \log n) < e^{n^{\frac{1}{k+1-\epsilon}}} < e^{\sqrt[k]{n}} \quad (0 < \epsilon < 1)$$

Corollary 2.3 *The following limit holds*

$$N_k(x)^{\frac{1}{\sqrt[k]{x}}} \rightarrow 1$$

Corollary 2.4 *The following limit holds*

$$p_k(n)^{\frac{1}{\sqrt[k]{n}}} \rightarrow 1$$

References

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