An Introduction to Order Prime Graph

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Abstract

Let Γ be a finite group. The order prime graph $OP(\Gamma)$ of a group Γ is a graph with $V(OP(\Gamma))=\Gamma$ and two vertices are adjacent in $OP(\Gamma)$ if and only if their orders are relatively prime in Γ. In this paper we obtain several properties of $OP(\Gamma)$, lower bounds and upper bounds on the number of edges of $OP(\Gamma)$ and characterise certain classes of order prime graph.

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1 INTRODUCTION

Throughout this paper, Γ denotes a group with identity e and G denotes a graph with order p and size q. Terms not defined here are used in the sense of Apostol1, Harary3 and Herstein4. Two integers a and b are said to be relatively prime if their greatest common divisor is 1 viz., $(a, b)=1$. Relatively prime integers play a significant role in both Analytic and Algebraic number theory. They motivated us to define order prime graph $OP(\Gamma)$, where Γ is a finite group. We hope that this definition will be a foundation stone for a new development in Algebraic Graph Theory. It is defined as a graph with $V(OP(\Gamma))=\Gamma$, where Γ is a finite group and two vertices are adjacent in $OP(\Gamma)$ if and only if their orders are relatively prime in Γ.
Definition 1.1. [3] Unicyclic graphs are graphs which are connected and have just one cycle.

Definition 1.2. [5] A graph $G$ in which each cycle of length at least four has a chord is called a triangulated graph.

Definition 1.3. [2] If $A$ is any subset of a group $G$, the subgroup of $G$ generated by $A$, $[A]$ is defined as the intersection of all subgroups of $G$ containing $A$. If $[A]=G$, then $A$ is said to be a generating set of $G$.

We use the following theorems.

Theorem 1.4. (CAUCHY)[4] If $p$ is a prime number and $p \mid o(G)$, then $G$ has an element of order $p$.

Theorem 1.5. [4] If $H$ and $K$ are finite subgroups of $G$ with orders $o(H)$ and $o(K)$ respectively, then $o(HK)=\frac{o(H)o(K)}{o(H\cap K)}$.

2 MAIN RESULTS

Definition 2.1. Let $\Gamma$ be a finite group. The order prime graph $OP(\Gamma)$ of a group $\Gamma$ is a graph with $V(OP(\Gamma))=\Gamma$ and two vertices $a$ and $b$ are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$. Here $o(a), o(b)$ respectively denote the orders of $a$ and $b$.

Example 2.2. Let $\Gamma=\mathbb{Z}_6$. Then $(\Gamma, +_6)$ is a group where $+_6$ is addition modulo 6. The graph $OP(\Gamma)$ is given in Figure 1.

We observe that $o(0) = 1$, $o(1) = 6$, $o(2) = 3$, $o(3) = 2$, $o(4) = 3$, $o(5) = 6$.

![Figure 1: OP(Γ)](image)

Proposition 2.3. Let $\Gamma$ be a group with $o(\Gamma)=n$. Then $\Delta(OP(\Gamma))=n-1$. 
**Proposition 2.4.** For any group $\Gamma$, $OP(\Gamma)$ is complete if and only if $o(\Gamma)=2$.

**Proof.** If $o(\Gamma)=2$, clearly $OP(\Gamma)\cong K_2$. Suppose $OP(\Gamma)$ is complete. If $o(\Gamma)\geq 3$ and every element in $\Gamma$ is a self-inverse element, then every element is of order 2 and so $OP(\Gamma)$ cannot be complete. If there exists at least one element which is not a self-inverse element, the element and its inverse have the same order in $\Gamma$ and hence non-adjacent in $OP(\Gamma)$. These contradictions show that $o(\Gamma)=2$. \qed

**Corollary 2.5.** Let $\Gamma$ be a group with $n\geq 3$ elements. Then $OP(\Gamma)$ cannot be regular.

**Proof.** The proof follows from proposition 2.3 and proposition 2.4. \qed

**Proposition 2.6.** For any group $\Gamma$, $G=OP(\Gamma)$ can never be a unicyclic graph.

**Proof.** Suppose $G$ is unicyclic. Since $\Delta(G)=n-1$ and $e$ is the vertex with degree $\Delta,G-e$ has exactly one edge $e'$. Let $e'=ab$. Then $(o(a),o(b))=1$. Now at least one of $a$ and $b$ should not be a self-inverse element since otherwise $o(a)=o(b)=2$. Let $a\neq a^{-1}$. Now $o(a)=o(a^{-1})$ and so $a^{-1}$, $b$ are also adjacent in $G-e$, which is a contradiction. \qed

**Theorem 2.7.** Let $\Gamma$ be a group with $n$ elements. $OP(\Gamma)\cong K_{1,n-1}$ if and only if $o(\Gamma)=p^\alpha$ where $p$ is a prime number and $\alpha\in \mathbb{Z}^+$. 

**Proof.** Let $\Gamma=\{e, a_1, ..., a_{n-1}\}$. Assume $o(\Gamma)=p^\alpha$ where $p$ is a prime number and $\alpha\in \mathbb{Z}^+$. Then $\forall i o(a_i)=p^k$ for some $k\in \{1, 2, ..., \alpha\}$. Now $a_i's$ are mutually non adjacent in $OP(\Gamma)$ and hence $OP(\Gamma)\cong K_{1,n-1}$.

Conversely, assume that $G=OP(\Gamma)\cong K_{1,n-1}$. Clearly $G-e$ is totally disconnected.

**Claim:** $o(\Gamma)=p^\alpha$.

Suppose $o(\Gamma)\neq p^\alpha$. Without loss of generality we shall assume that $o(\Gamma)=p_1^{k_1}p_2^{k_2}$ where $p_1$, $p_2$ are prime numbers and $k_1, k_2 \in \mathbb{Z}^+$. Since $p_1|o(\Gamma)$ and $p_2|o(\Gamma)$, by Theorem 1.4, $\Gamma$ has elements of order $p_1$ and $p_2$. Let $a, b$ be elements of $\Gamma \ni o(a)=p_1$ and $o(b)=p_2$. Hence $a, b$ are adjacent in $G-e$, which is a contradiction. \qed

**Corollary 2.8.** Let $\Gamma$ be a group with $n$ elements. $OP(\Gamma)$ is a tree if and only if $o(\Gamma)=p^\alpha$ where $p$ is a prime number and $\alpha\in \mathbb{Z}^+$. 

**Proof.** The proof follows from theorem 2.3 and theorem 2.5. \qed
Proof. Since \( \Delta(OP(\Gamma)) = n - 1 \), \( OP(\Gamma) \) is connected. Hence \( OP(\Gamma) \) is a tree \( \iff OP(\Gamma) \cong K_{1, n-1} \iff o(\Gamma) = p^\alpha \) where \( p \) is prime and \( \alpha \in \mathbb{Z}^+ \).

**Proposition 2.9.** Let \( \Gamma \) be a cyclic group. Then \( OP(\Gamma) \) has at least two pendent vertices.

**Proof.** If \( \Gamma \) is a cyclic group of order two, then \( OP(\Gamma) \cong K_2 \) and hence the proof follows. Suppose \( \Gamma \) is a cyclic group of order greater than two. Let \( a \) be the generator of \( \Gamma \). Clearly \( o(a) = o(\Gamma) \). Since \( o(\Gamma) \neq 2 \), \( a^{-1} \neq a \) is also a generator of \( \Gamma \) and so \( o(a) = o(a^{-1}) = o(\Gamma) \). Since the order of any element of \( \Gamma \) divides the order of \( \Gamma \), \( a \) and \( a^{-1} \) are adjacent to identity element only. Thus \( OP(\Gamma) \) has at least two pendent vertices. \( \square \)

**Remark 2.10.** \( OP(\Gamma) \) need not always be a triangulated graph. Let \( \Gamma_1 \cong S_3 \). \( OP(\Gamma_1) \) is given in Figure 2. The graph induced by the set \( \{(12), (23), (123), (132)\} \) in \( OP(\Gamma_1) \) is \( C_4 \) and so \( OP(\Gamma_1) \) is not a triangulated graph. But the graph given in example 2.2 is a triangulated graph.

![Figure 2: \( OP(\Gamma_1) \)]

**Theorem 2.11.** If \( \Gamma_1, \Gamma_2 \) are two groups such that \( \Gamma_1 \cong \Gamma_2 \), then \( OP(\Gamma_1) \cong OP(\Gamma_2) \). But the converse is not true.

**Proof.** Let \( \Phi \) be an isomorphism of \( \Gamma_1 \) onto \( \Gamma_2 \).

Now, \( a \) and \( b \) are adjacent in \( OP(\Gamma_1) \)

\[ \iff (o(a), o(b)) = 1 \]

\[ \iff (o(\Phi(a)), o(\Phi(b))) = 1 \]

\[ \iff \Phi(a) \text{ and } \Phi(b) \text{ are adjacent in } OP(\Gamma_2) \]. Thus \( OP(\Gamma_1) \cong OP(\Gamma_2) \).

Conversely, consider the groups \( (\Gamma_1, .), (\Gamma_2, \Delta) \) where \( \Gamma_1 = \{ 1, -1, i, -i \} \), \( \Gamma_2 = \{ \Phi, \{ a \}, \{ b \}, \{ ab \} \} \) and \( ., \Delta \) respectively denote multiplication and symmetric difference. Since \( o(OP(\Gamma_1)) = o(OP(\Gamma_2)) = 2^2 \), by Theorem 2.7, \( OP(\Gamma_1) \cong OP(\Gamma_2) \cong K_{1, 3} \). But \( \Gamma_1 \) has an element of order 4 whereas \( \Gamma_2 \) does not have an element of order 4. Thus \( \Gamma_1 \not\cong \Gamma_2 \). \( \square \)

**Theorem 2.12.** Let \( \Gamma \) be a group. Then \( Aut(\Gamma) \subseteq Aut(OP(\Gamma)) \)
**Proof.** Let $\Gamma$ be a group. $g \in \text{Aut}(\Gamma)$ implies $g$ is an isomorphism of $\Gamma$ onto $\Gamma$. Now, $a$ and $b$ are adjacent in $\text{OP}(\Gamma)$ if and only if $(o(a), o(b)) = 1$.

**Remark 2.13.** The converse of theorem 2.12 is not true. Consider the group $(\Gamma = \mathbb{Z}_5, +_5)$. Here $\text{OP}(\Gamma) \cong K_{1,4}$. Define $g: \text{OP}(\Gamma) \to \text{OP}(\Gamma)$ such that $g(0) = 0$, $g(1) = 2$, $g(2) = 3$, $g(3) = 4$, $g(4) = 1$. Clearly $g$ is an automorphism of $\text{OP}(\Gamma)$. But $g(1 + 5) = g(3) = 4$ and $g(1 + 5) = 2 + 5 = 3 = 0$. Hence $g(1 + 5) \neq g(1) + 5g(2)$ so that $g$ is not an automorphism of $\Gamma$.

**Theorem 2.14.** Let $\Gamma$ be a group. Suppose $\text{OP}(\Gamma)$ has two adjacent elements $a, b$ such that $o(a)o(b) = o(\Gamma)$. Then the set $\{a, b\}$ is a generating set of $\Gamma$.

**Proof.** Given $a, b$ are two adjacent elements in $\text{OP}(\Gamma)$ and so $(o(a), o(b)) = 1$. Let $o(a) = m, o(b) = n$. Let $H = \langle a \rangle$ and $K = \langle b \rangle$ be two subgroups of $\Gamma$. We have $o(H) = m$ and $o(K) = n$. Since $(m, n) = 1$, $H \cap K = \{e\}$, $o(H \cap K) = 1$. By Theorem 1.5, $o(HK) = \frac{o(H)o(K)}{o(H \cap K)} = o(H)o(K) = mn = o(\Gamma)$. Hence $o(HK) = o(\Gamma)$. Since $HK$ is a subset of $\Gamma$, $HK = \Gamma$. $a, b \in \Gamma \Rightarrow \{\{a, b\}\}$ is a subset of $\Gamma$. Now $x \in HK \Rightarrow x = anb \Rightarrow x \in \{\{a, b\}\}$. $\Gamma = HK \subseteq \{\{a, b\}\}$ and hence $\{a, b\}$ is a generating set of $\Gamma$.

**Theorem 2.15.** Let $\Gamma$ be an abelian group. Let $X \subseteq \Gamma$ be such that the graph induced by $X$ is complete in $\text{OP}(\Gamma)$ and the product of the order of all elements of $X$ is same as $o(\Gamma)$. Then $X$ is a generating set of $\Gamma$.

**Proof.** Let $X = \{a_1, a_2, ..., a_k\} \subseteq \Gamma$. By assumption, $X$ is complete in $\text{OP}(\Gamma)$ and so $(o(a_i), o(a_j)) = 1 \forall i \neq j$. For $1 \leq i \leq k$, consider the subgroups of $\Gamma$, $H_i = \langle a_i \rangle$. Since $(o(H_i), o(H_j)) = 1 \forall i \neq j$, $H_i \cap H_j = \{e\} \forall i \neq j$. Now $o(H_1H_2 ... H_k) = \frac{o(H_1)o(H_2)}{o(H_1 \cap H_2)}$. Since $(H_1 \cap H_2)(H_1 \cap H_3)...(H_1 \cap H_k) \subseteq H_1 \cap H_2 ... H_k$, $o(H_1H_2 ... H_k) \geq \min \frac{o(H_1)o(H_2)}{1} = o(H_1)o(H_2 ... H_k)$. Proceeding like this we get, $o(H_1H_2 ... H_k) \geq o(H_1)o(H_2)...o(H_k) = o(\Gamma)$.

**Theorem 2.16.** Let $\Gamma$ be a group with $o(\Gamma) = p_{1}^{n_1}p_{2}^{n_2}...p_{k}^{n_k}$ where $p_i$s are prime numbers and $n_i \in \mathbb{Z}^+$ ($1 \leq i \leq k$). Then $\text{OP}(\Gamma)$ is a complete $(k + 1)$-partite graph if and only if $o(a) = p_i^{e} \forall a \in \Gamma - e$, $1 \leq i \leq k$ and $1 \leq j \leq n_i$. 
Proof. Let \( \Gamma \) be a group with \( o(\Gamma) = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \) where \( p_i \)'s are prime numbers and \( n_i \in \mathbb{Z}^+ (1 \leq i \leq k) \). For every \( 1 \leq i \leq k \), define the set
\[
V_i = \{a \in \Gamma - e/o(a) = p_i^j, 1 \leq j \leq n_i\} \text{ and } V_0 = \{e\}. \]
Theorem 1.4 guarantees that \( V_i \neq \emptyset \). Also \( V_i \cap V_j = \emptyset \forall i \neq j \) and \( \bigcup_{i=0}^{k} V_i = \Gamma \). No two elements in \( V_i \) are adjacent and \( (p_i^r, p_j^s) = 1 \forall i \neq j, 1 \leq i \leq n_i \) and \( 1 \leq s \leq n_j \). Hence \( OP(\Gamma) \) is a complete \((k + 1)\)-partite graph with parition \( V_0, V_1, ..., V_k \).
Conversely, assume that \( OP(\Gamma) \) is a complete \((k + 1)\)-partite graph. Clearly one partition is the single element \( e \). Without loss of generality we shall assume that it is the first parition.According to theorem 1.4 \( \Gamma \) has elements \( a_i \) such that \( o(a_i) = p_i^j, 1 \leq i \leq k \). No two \( a_i \)'s can lie in the same partition of \( OP(\Gamma) \).
Let us suppose that \( a_i \) belongs to \((i + 1)\)th partition. Each \((i + 1)\)th-partition has the elements of order \( mp_i \) where \( m \in \mathbb{Z}^+ \).
Claim: \( o(a) = p_i^j \forall a \in \Gamma - e, 1 \leq j \leq n_i \) and \( 1 \leq i \leq k \).
If not, \( \exists \) an element \( b \in \Gamma - e \exists o(b) \neq p_i^j \). Without loss of generality we shall assume that \( o(b) = p_i p_2 \). Hence \( b \) lies in either \( 2^{nd} \) or \( 3^{rd} \) partition and accordingly \( a_2 b \) or \( a_1 b \) is not an edge. This is a contradiction as \( OP(\Gamma) \) is a complete \((k + 1)\)-partite graph.

\[\boxed{\text{Theorem 2.17. Let } \Gamma \text{ be a group with } o(\Gamma) = n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}. \text{ Let } q \text{ be the number of edges of the graph } G = OP(\Gamma). \text{ Then } n - 1 + \frac{1}{2} [m_1^2 - (m_1^2 + ... + m_k^2)] \leq q \leq \frac{1}{2} [n^2 - n + m - (m_1^2 + ... + m_k^2)]\}
\]

Moreover, these bounds are sharp.

Proof. Clearly the identity element is adjacent to remaining \((n - 1)\) elements, and each of the \( m_i \) elements are adjacent to all the \( m_j \) elements whenever \( i \neq j \). Hence the minimum number of edges in \( G \) equals
\[
n - 1 + \frac{1}{2} [m_1(m - m_1) + m_2(m - m_2) + ... + m_k(m - m_k)]
\]
\[
= n - 1 + \frac{1}{2} [m_1^2 - m_1^2 + ... + m_k^2 - m_k^2]
\]
\[
= n - 1 + \frac{1}{2} [m_1 + m_2 + ... + m_k] - (m_1^2 + ... + m_k^2)]
\]
\[
= n - 1 + \frac{1}{2} [m^2 - (m_1^2 + ... + m_k^2)]. \text{ Also these } m_i \text{ elements are mutually non-adjacent. Hence the maximum number of edges of } G \text{ equals}
\]
\[
\begin{align*}
\binom{n}{2} - \left\{ \binom{m_1}{2} + \binom{m_2}{2} + ... + \binom{m_k}{2} \right\}
\end{align*}
\]
\[
= \binom{n - 1}{2} - \left\{ \binom{m_1(m_1 - 1)}{2} + ... + \binom{m_k(m_k - 1)}{2} \right\}
\]
\[
= \frac{1}{2} [n^2 - n - \{m_1^2 + ... + m_k^2 - (m_1 + ... + m_k)]
\]
\[
= \frac{1}{2} [n^2 - n + m - (m_1^2 + ... + m_k^2)]. \text{ Hence } n - 1 + \frac{1}{2} [m^2 - (m_1^2 + ... + m_k^2)]
\]
\[
\leq q \leq \frac{1}{2} [n^2 - n + m - (m_1^2 + ... + m_k^2)].
\]
Moreover, the following examples exhibit that the bounds are sharp. Consider \( \Gamma \) given in example 2.2. Here \( n = 6, m_1 = 2, m_2 = 1 \) and \( m = 3 \). The number of
edges in $OP(\Gamma) = (6-1) + \frac{1}{2}[3^2 - (2^2 + 1^2)] = 5 + 2 = 7$. If one considers $OP(\Gamma_1)$ given in the Figure 2, then $n=6$, $m_1=3$, $m_2=2$ and $m=5$. The number of edges in $OP(\Gamma_1) = \frac{1}{2}[6^2 - 6 + 5 - (3^2 + 2^2)] = \frac{1}{2}[22] = 11$. \hfill \Box

**Corollary 2.18.** Let $\Gamma$ be a group with $o(\Gamma) = p_1^{n_1}p_2^{n_2}\ldots p_k^{n_k}$. Let $q$ be the number of edges of the graph $OP(\Gamma)$. Then the upper bound and lower bound of $q$ in (1) are equal if and only if $OP(\Gamma)$ is a complete $(k+1)$-partite graph.

**Proof.** Assume that the upper bound and lower bound of $q$ in (1) are equal. Thus we have $n-1 + \frac{1}{2}[m^2 - (m_1^2 + \ldots + m_k^2)] = \frac{1}{2}[n^2 - n + m - (m_1^2 + \ldots + m_k^2)]$

$\Rightarrow 2n - 2 + m^2 = n^2 - n + m$

$\Rightarrow m^2 - m = n(n - 2)(n - 1)$

$\Rightarrow m^2 - m - [(n - 1)(n - 2)] = 0$

$\Rightarrow (m + (n - 2))(m + (1 - n)) = 0$

$\Rightarrow m = -(n - 2)\text{ or } m = n - 1$. But $m = -(n - 2)$ is not possible. Hence $m = n - 1$.

This means $o(a) = p_i^j \forall a \in \Gamma - e$, $1 \leq j \leq n_i$ and $1 \leq i \leq k$. Thus by Theorem 2.16 $OP(\Gamma)$ is a complete $(k + 1)$-partite graph. Conversely assume that $OP(\Gamma)$ is a complete $(k + 1)$-partite graph. By Theorem 2.16 $o(a) = p_i^j \forall a \in \Gamma - e$, $1 \leq j \leq n_i$ and $1 \leq i \leq k$. Hence $m = n - 1$. Now lower bound of $q$

$= n - 1 + \frac{1}{2}[m_1^2 + \ldots + m_k^2]

= n - 1 + \frac{1}{2}[(n - 1)^2 - (m_1^2 + \ldots + m_k^2)]

= n - 1 + \frac{1}{2}[n^2 - 2n + 1 - (m_1^2 + \ldots + m_k^2)]

= \frac{1}{2}[2n - 2 + n^2 - 2n + 1 - (m_1^2 + \ldots + m_k^2)]

= \frac{1}{2}[n^2 - 1 - (m_1^2 + \ldots + m_k^2)].$

Upper bound of $q = \frac{1}{2}[n^2 - n + m - (m_1^2 + \ldots + m_k^2)]$

$= \frac{1}{2}[n^2 - n + n - 1 - (m_1^2 + \ldots + m_k^2)]$

$= \frac{1}{2}[n^2 - 1 - (m_1^2 + \ldots + m_k^2)]$. Hence the lower bound of $q$ equals the upper bound of $q$. \hfill \Box

**References**


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