Iterative Methods for a Class of Linear Operator Equations

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Abstract

This paper presents a two step iterative method for the solution of a linear operator equation, where the operator admits a positive definite and m-accretive splitting. The iterations alternate between the positive definite and m-accretive part of the operator. Theoretical analysis shows that the method converges unconditionally to the solution of the equation. Additionally, the analysis of a successive overrelaxation acceleration of this method is provided. The convergence of the proposed methods are illustrated by a numerical example in which an integro-differential problem of transport theory is considered.

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1 Introduction and Preliminaries

Linear equations with operators admitting positive definite and m-accretive splitting often arise in physics and in engineering. They find many applications in particle transport, radiative transfer, diffusion convection etc...

Let us consider a Hilbert space \(H\) with inner product \((..)\) and norm \(\|\cdot\|\) and let \(T\) be a linear operator on \(H\) with domain \(\mathcal{D}(T)\) and range \(\mathcal{R}(T) = H\).
We denote by $I$, the identity operator. Suppose that we need to solve in $\mathcal{D}(T)$, the following problem

$$Tu = q,$$

(1)

where $q \in H$ is given and $u \in \mathcal{D}(T)$ is the unknown.

We assume that the operator $T$ admits the following splitting:

$$T = P + A,$$

(2)

where $P$ is a bounded positive definite operator and $A$ is a m-Accretive operator. Therefore, the operator $T$ is positive definite and equation (1) admits a unique solution in $H$.

We introduce in this paper a two-step iteration method linked to the Positive definite and m-Accretive Splitting (2) of operator $T$. We investigate the convergence of this iterative method. Theoretical analysis shows that this iterative method converges to the solution of equation (1). An analysis of a successive overrelaxation acceleration of this iteration method is provided. We obtain similar results as in the case of finite dimensional linear systems with coefficient matrices possessing Property A [7, 12]. The proposed methods are illustrated by a numerical example in which an integro-differential problem of transport theory is considered.

We consider in $\mathcal{D}(T)$ the norm

$$\|u\|^2_{\mathcal{D}(T)} = \|u\|^2 + \|Au\|^2.$$  

(3)

**Proposition 1.1** Let $\alpha$ be a positive constant. The functional $\rho_{A(\alpha)}$ defined on $\mathcal{D}(T)$ by

$$\rho_{A(\alpha)}(u) = \|(\alpha I + A)u\|,$$

(4)

is a norm on $\mathcal{D}(T)$ equivalent to $\|\cdot\|_{\mathcal{D}(T)}$.

**Proof.** Let $\alpha$ be a positive constant. It can be easily seen from the linearity of the operator $\alpha I + A$ and the properties of the norm $\|\cdot\|$ that $\forall u, v \in \mathcal{D}(T)$ and $\beta \in \mathbb{C}$

$$\rho_{A(\alpha)}(u + v) \leq \rho_{A(\alpha)}(u) + \rho_{A(\alpha)}(v) \quad \text{and} \quad \rho_{A(\alpha)}(\beta u) = |\beta|\rho_{A(\alpha)}(u).$$

Moreover, since $A$ is m-accretive, $\alpha I + A$ is positive definite and $\rho_{A(\alpha)}(u) = 0$ if and only if $u = 0$. It then follows that $\rho_{A(\alpha)}$ is a norm on $\mathcal{D}(T)$.

Let $u \in \mathcal{D}(T)$. We have

$$(\rho_{A(\alpha)}(u))^2 = \alpha^2\|u\|^2 + \|Au\|^2 + 2\alpha(u, Au).$$
Since \((Au, u) \geq 0 (A \text{ is accretive})\) and \((Au, u) \leq \frac{\|u\|^2 + \|Au\|^2}{2}\), we have
\[
\min\{\alpha^2, 1\} \|u\|^2_D(T) \leq (\rho_{A(\alpha)}(u))^2 \leq (\alpha + 1)^2 \|u\|^2_D(T).
\]
It follows that, the norms \(\|\cdot\|_D(T)\) and \(\rho_{A(\alpha)}\) are equivalent in \(D(T)\). \(\square\)

In the following, the norm \(\rho_{A(\alpha)}\) is denoted by \(\|\cdot\|_{A(\alpha)}\).

The paper is organized as follows. In Section 2, we present the two step iterative methods and the convergence analysis. Section 3 deals with the analysis of a successive overrelaxation acceleration of the method. The numerical results are presented in Section 4. Some concluding remarks are given in section 5.

## 2 The iteration method

We present in this section an iterative method for the solution of the problem (1). It is a two-step iteration method relies on the splitting (2) and alternating between the positive definite and the m-accretive parts of the operator \(T\).

Let \(\alpha\) be a positive constant. The following two-step splitting is obtained from (2):
\[
\begin{align*}
T &= (\alpha I + P) - (\alpha I - A) \\
T &= (\alpha I + A) - (\alpha I - P)
\end{align*}
\] (5)

The two-step splitting (5) leads to the following Positive definite and m-Accretive Splitting (PAS) iteration method:

Given an initial guess \(u^{(0)} \in D(T)\), for \(k = 0, 1, \ldots\) until \(\left\{u^{(k)}\right\}\) converges, calculate
\[
\begin{align*}
(\alpha I + P)u^{(k+\frac{1}{2})} &= (\alpha I - A)u^{(k)} + q \\
(\alpha I + A)u^{(k+1)} &= (\alpha I - P)u^{(k+\frac{1}{2})} + q
\end{align*}
\] (6)

From equation (6), we deduce that \(u^{k+1}\) satisfies
\[
(\alpha I + A)u^{(k+1)} = M(\alpha)(\alpha I + A)u^{(k)} + N(\alpha)q,
\] (7)
where
\[
M(\alpha) = V(\alpha)U(\alpha) \text{ and } N(\alpha) = 2\alpha(\alpha I + P)^{-1};
\] (8)
with
\[
V(\alpha) = (\alpha I - P)(\alpha I + P)^{-1} \text{ and } U(\alpha) = (\alpha I - A)(\alpha I + A)^{-1}.
\] (9)

Therefore, the exact solution \(u^*\) of the problem (1) verifies
\[
\|u^{(k+1)} - u^*\|_{A(\alpha)} \leq \|M(\alpha)\|\|u^{(k)} - u^*\|_{A(\alpha)}.
\] (10)
It is well known that the iteration method (6) converge (in the sense of the norm $\| \cdot \|_{A(\alpha)}$) if the operator $M(\alpha)$ is a strict contraction or equivalently, if
\[
\|(M(\alpha))\| < 1. \tag{11}
\]

**Proposition 2.1** Convergence of the PAS iteration method.

Let $\alpha$ be a positive constant. The norm $\|(M(\alpha))\|$ of the iteration operator $M(\alpha)$ verifies
\[
\|(M(\alpha))\| < 1. \tag{12}
\]

Therefore it holds that, the PAS iteration converges to the unique solution $u^* \in D(T)$ of the problem (1).

**Proof.** The proof of this proposition is based on the following lemmas.

**Lemma 2.2** If $X$ is a m-accretive operator in the Hilbert space $H$, then for $\alpha > 0$,
\[
\|((\alpha I - X)(\alpha I + X)^{-1})\| \leq 1. \tag{13}
\]

**Proof. (of Lemma 2.2)** Let $\alpha > 0$. If the operator $X$ is m-accretive then the operator $(\alpha I + X)^{-1}$ is bounded and, for $u \in D(X)$, $(Xu, u) \geq 0$. Moreover, we have
\[
\|((\alpha I - X)u\|^2 - \|(\alpha I + X)u\|^2 = -4\alpha(Xu, u) \leq 0.
\]
Taking $u = (\alpha I + X)^{-1}\varphi$, $(\varphi \in H)$, we obtain
\[
\|((\alpha I - X)(\alpha I + X)^{-1}\varphi\|^2 \leq \|\varphi\|^2.
\]
It follows that $\|((\alpha I - X)(\alpha I + X)^{-1}\| \leq 1$. □

**Lemma 2.3** If $X$ is a positive definite operator in the Hilbert space $H$, then for $\alpha > 0$,
\[
\|((\alpha I - X)(\alpha I + X)^{-1}\| < 1. \tag{14}
\]

**Proof. (of Lemma 2.3)** Let $\alpha > 0$. If the operator $X$ is positive definite, then for $u \in D(X)$, $(Xu, u) > 0$. Moreover, we have
\[
\|((\alpha I - X)u\|^2 - \|(\alpha I + X)u\|^2 = -4\alpha(Xu, u) < 0.
\]
Taking $u = (\alpha I + X)^{-1}\varphi$, $(\varphi \in H)$, we obtain
\[
\|((\alpha I - X)(\alpha I + X)^{-1}\varphi\|^2 < \|\varphi\|^2.
\]
It follows that  \( \| (\alpha I - X)(\alpha I + X)^{-1} \| < 1 \). □

We have
\[
\| M(\alpha) \| \leq \| U(\alpha) \| \| V(\alpha) \| .
\]

Since \( P \) is positive definite and \( A \) is \( m \)-accretive, we deduce from Lemma 2.2 and Lemma 2.3 that
\[
\| U(\alpha) \| = \| (\alpha I - A)(\alpha I + A)^{-1} \| \leq 1
\]
and
\[
\| V(\alpha) \| = \| (\alpha I - P)(\alpha I + P)^{-1} \| < 1.
\]

It then follows that
\[
\| M(\alpha) \| \leq \| U(\alpha) \| \| V(\alpha) \| < 1. □
\]

From the equivalence between the norms \( \| . \|_{D(T)} \) and \( \| . \|_{A(\alpha)} \) follows the convergence with respect to the norm \( \| . \|_{D(T)} \) in \( D(T) \). Additionally, since
\[
\| u^{(k+1)} - u^* \| \leq \| u^{(k+1)} - u^* \|_{D(T)}; \quad k = 0, 1, 2, \cdots , \tag{15}
\]
we have
\[
\lim_{k \to +\infty} \| u^{(k)} - u^* \| = \lim_{k \to +\infty} \| u^{(k)} - u^* \|_{D(T)} = 0. \tag{16}
\]

Thus the sequence \( u^{(k)} \) converges in \( D(T) \) with respect to the norm \( \| . \| \).

Each step of the PAS iterative method is constituted of two-half steps which require finding solutions of linear equations with operators \( (\alpha I + P) \) and \( (\alpha I + A) \). Exact solutions of these equations are generally not available. These linear operator equations can be solved approximately using appropriate methods with respect to the properties of each operators. This results in the inexact Positive definite and \( m \)-Accretive splitting (IPAS) iteration for solving the linear operator equation (1).

### 3 Successive overrelaxation (SOR) acceleration of the PAS Iteration Method.

The following fixed point equation can be derived from the definition of the PAS iteration (6):
\[
\begin{align*}
(\alpha I + P)u_1 &= (\alpha I - A)u_2 + q \\
(\alpha I + A)u_2 &= (\alpha I - P)u_1 + q . \tag{17}
\end{align*}
\]

In the operator form, the system (17) reads
\( T(\alpha)u = q, \) \hspace{1cm} (18)

where the matrix of operators \( T(\alpha) \) and the vector functions \( u \) and \( q \) are defined as follows:

\[
T(\alpha) = \begin{pmatrix}
(\alpha I + P) & -(\alpha I - A) \\
-(\alpha I - P) & (\alpha I + A)
\end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} q \\ q \end{pmatrix}.
\]

**Proposition 3.1** Let \( \alpha \) be a positive constant. The solution of linear operator equation (18) exists in \( \mathcal{D}(T) \times \mathcal{D}(T) \) and its unique.

**Proof.** For \( \alpha > 0 \), the operators \((\alpha I + A)\) and \((\alpha I + P)\) are positive definite. It follows that \((\alpha I + A)^{-1}\) and \((\alpha I + P)^{-1}\) exist and it holds that

\[
T(\alpha) = \begin{pmatrix}
I \\
-(\alpha I - P)(\alpha I + P)^{-1} & I
\end{pmatrix} \begin{pmatrix}
(\alpha I + P) & -(\alpha I - A) \\
0 & A(\alpha)
\end{pmatrix}
\]

with

\[
A(\alpha) = (\alpha I + A) - (\alpha I - P)(\alpha I + P)^{-1}(\alpha I - A)
\]

\[
= [I - (\alpha I - P)(\alpha I + P)^{-1}(\alpha I - A)(\alpha I + A)^{-1}](\alpha I + A)
\]

\[
= (I - M(\alpha))(\alpha I + A).
\]

Since \( \|M(\alpha)\| < 1 \), \((I - M(\alpha))^{-1}\) exists and \(A(\alpha)\) is invertible. It then follows that the matrix operator \(T(\alpha)\) is invertible. \(\square\)

We consider in \( H \times H \) the norm \( \|.|.\| \) given for \( u = (u_1, u_2)^t \in H \times H \) by

\[
\|u\|^2 = \|u_1\|^2 + \|u_2\|^2. \hspace{1cm} (19)
\]

**Proposition 3.2** Let \( \alpha \) be a positive constant. If \( u^* \) is the exact solution of problem (1), then \( u^* = \begin{pmatrix} u^*_1 \\ u^*_2 \end{pmatrix} \) is the exact solution of equation (18). Conversely, if \( u^* = \begin{pmatrix} u^*_1 \\ v^*_2 \end{pmatrix} \) is the exact solution of equation (18) then \( u^* = v^* \) and, \( u^* \) is the exact solution of problem (1).

**Proof.** Let \( u^* \) be the solution of problem (1). We have \( Tu^* = q \) and

\[
T(\alpha)\begin{pmatrix} u^*_1 \\ u^*_2 \end{pmatrix} = \begin{pmatrix} (\alpha I + P)u^* - (\alpha I - A)u^* \\ -(\alpha I - P)u^* + (\alpha I + A)u^* \end{pmatrix} = \begin{pmatrix} Tu^*_1 \\ Tu^*_2 \end{pmatrix} = q.
\]

Let \( u^* = \begin{pmatrix} u^* \\ v^* \end{pmatrix} \) be the solution of equation (18). The functions \( u^* \) and \( v^* \) satisfy

\[
(\alpha I + P)u^* - (\alpha I - A)v^* = q, \hspace{1cm} (20)
\]
\[-(\alpha I - P)u^* + (\alpha I + A)v^* = q.\]  \hspace{1cm} (21)

By subtracting (21) from (20), we have

\[2\alpha u^* - 2\alpha v^* = 0.\]

It follows that \(v^* = u^*\). Substituting \(v^*\) in (20), we obtain \(Tu^* = q.\) \(\square\)

Let \(P(\alpha)\) be the matrix operator defined in \(D(T) \times D(T)\) by:

\[P(\alpha) = \begin{pmatrix} (\alpha I + P) & 0 \\ 0 & (\alpha I + A) \end{pmatrix}.\]  \hspace{1cm} (22)

The preconditioning of the system (18) from the right by \([P(\alpha)]^{-1}\) leads to the following system

\[T_1(\alpha)u = q\]  \hspace{1cm} (23)

where the matrix of operator \(T_1(\alpha)\) reads

\[T_1(\alpha) = \begin{pmatrix} I & -A_1(\alpha) \\ -P_1(\alpha) & I \end{pmatrix},\]  \hspace{1cm} (24)

with \(A_1(\alpha) = (\alpha I - A)(\alpha I + A)^{-1}\) and \(P_1(\alpha) = (\alpha I - P)(\alpha I + P)^{-1}\). The solution \(v^*\) of problem (18) reads

\[v^* = [P(\alpha)]^{-1}u^*,\]  \hspace{1cm} (25)

where \(u^*\) is solution of (23).

Since all the operators of the matrix \(T_1(\alpha)\) are bounded in \(H\), \(T_1(\alpha)\) is bounded in \(H \times H\).

Given an initial guest \(u^{(0)}\) in \(H \times H\), the Jacobi iteration for the solution of (23) reads

\[u^{(k+1)} = J(\alpha)u^{(k)} + q,\]  \hspace{1cm} (26)

where

\[J(\alpha) = \begin{pmatrix} 0 & A_1(\alpha) \\ P_1(\alpha) & 0 \end{pmatrix}.\]  \hspace{1cm} (27)

The corresponding SOR iteration with the relaxation parameter \(\omega\), is

\[u^{(k+1)} = L_\omega(\alpha)u^{(k)} + q_\omega(\alpha),\]  \hspace{1cm} (28)

where

\[L_\omega(\alpha) = \begin{pmatrix} (1 - \omega)I & \omega A_1(\alpha) \\ \omega(1 - \omega)P_1(\alpha) & (1 - \omega)I + \omega^2 M(\alpha) \end{pmatrix}.\]  \hspace{1cm} (29)

The choice of \(\omega = 1\) in the method (28) results in the Gauss-Seidel iteration for solving the system (23).
Proposition 3.3  The Jacobi and Gauss-Seidel iteration methods for the solution of (23) converge.

Proof. For \( u = (u_1, u_2)^t \in H \times H \),
\[
\|J(\alpha)u\|^2 = \|A_1(\alpha)u_2\|^2 + \|P_1(\alpha)u_1\|^2 < \|u_2\|^2 + \|u_1\|^2 = \|u\|^2.
\]
It follows that \( \|J(\alpha)\| < 1 \) and the Jacobi iteration method (26) converges.

The Gauss-Seidel iteration is equivalent to the following iteration:
\[
\begin{align*}
&u_1^{(k+1)} = A_1(\alpha)P_1(\alpha)u_1^{(k)} + (A_1(\alpha) + I)q \\
&u_2^{(k+1)} = P_1(\alpha)A_1(\alpha)u_2^{(k)} + (P_1(\alpha) + I)q.
\end{align*}
\]

Let \( u^* = (u_1^*, u_2^*)^t \) be the exact solution to (23), we have
\[
\begin{align*}
&u_1^{(k+1)} - u_1^* = A_1(\alpha)P_1(\alpha)[u_1^{(k)} - u_1^*] \\
&u_2^{(k+1)} - u_2^* = P_1(\alpha)A_1(\alpha)[u_2^{(k)} - u_2^*]
\end{align*}
\]
and
\[
\begin{align*}
&\|u_1^{(k+1)} - u_1^*\| \leq \|A_1(\alpha)P_1(\alpha)\| \|u_1^{(k)} - u_1^*\| < \|u_1^{(k)} - u_1^*\|, \\
&\|u_2^{(k+1)} - u_2^*\| \leq \|P_1(\alpha)A_1(\alpha)\| \|u_2^{(k)} - u_2^*\| < \|u_2^{(k)} - u_2^*\|.
\end{align*}
\]
Therefore, the exact solution \( u^* \) verifies
\[
\|u^{(k+1)} - u^*\| < \|u^{(k)} - u^*\|.
\]
It follows that the Gauss-Seidel iteration method converges. \( \square \)

Proposition 3.4  Let \( R_1 \) and \( R_2 \) be two operators defined from \( H \) to \( H \). If \( R_1 \) and \( R_2 \) are bounded, then for \( \gamma \neq 0 \), the operators
\[
R = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix} \quad \text{and} \quad R(\gamma) = \begin{pmatrix} 0 & \gamma R_1 \\ \frac{1}{\gamma} R_2 & 0 \end{pmatrix}
\]
have the same spectrum.

Proof. We have to prove that if \( \lambda \notin \sigma(R) \), then \( \lambda \notin \sigma(R(\gamma)) \) and conversely.

Let \( \gamma \neq 0 \) and \( \lambda \in \mathbb{C} \). The operators \( R \) and \( R(\gamma) \) satisfy
\[
R - \lambda I = X(\gamma)(R(\gamma) - \lambda I)X^{-1}(\gamma); \\
R(\gamma) - \lambda I = X^{-1}(\gamma)(R - \lambda I)X(\gamma).
\]
where \( X(\gamma) = \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} \). The operator \( X(\gamma) \) is bounded and has a bounded inverse. If \( \lambda \notin \sigma(R) \), the operator \( (R - \lambda I)^{-1} \) is bounded. For \( v \in H \times H \).

The solution \( u^* \) of the linear equation
\[
(R(\gamma) - \lambda I)u = v
\]
is given by
\[ u^* = X^{-1}(\alpha)(R - \lambda I)^{-1}X(\alpha)v. \]
Since \( X(\gamma) \), \((R - \lambda I)^{-1}\) and \( X^{-1}(\gamma)\) are bounded operators, \( u^* \) verifies
\[ \|u^*\| \leq C\|v\|, \]
where \( C \) is a constant independent of \( v \).

Conversely, if \( \lambda \notin \sigma(R) \), the operator \((R(\gamma - \lambda I)^{-1})\) is bounded. For \( v \in H \times H \). The solution \( u^* \) of the linear equation
\[ (R - \lambda I)u = v \]
is given by
\[ u^* = X^{-1}(\alpha)(R(\gamma) - \lambda I)^{-1}X(\alpha)v; \]
and verifies
\[ \|u^*\| \leq C\|v\|, \]
where \( C \) is a constant independent of \( v \). It follows that \( \lambda \notin \sigma(R) \).

**Proposition 3.5** Assume that \( \omega \neq 0 \). If \( \lambda \in \sigma(L_\omega(\alpha)) \setminus \{0\} \) and if \( \tau \) satisfies
\[ (\lambda + \omega - 1)^2 = \lambda\omega^2\tau^2, \tag{30} \]
then \( \tau \in \sigma(J(\alpha)) \). Conversely, if \( \tau \in \sigma(J(\alpha)) \) and if \( \lambda \) satisfies \( (30) \), then \( \lambda \in \sigma(L_\omega(\alpha)) \).

**Proof.** For \( \omega \neq 0 \), we have to prove that:

1. If \( \lambda \notin \sigma(L_\omega(\alpha)) \) and if \( \tau \) satisfies \( (30) \), then \( \tau \notin \sigma(J(\alpha)) \);

2. If then \( \tau \notin \sigma(J(\alpha)) \) and if \( \lambda \) satisfies \( (30) \), then \( \lambda \notin \sigma(L_\omega(\alpha)) \).

The operators \( J(\alpha) \) and \( L_\omega(\alpha) \) can be written as
\[ L_\omega(\alpha) = (I + \omega R_1(\alpha))^{-1}((1 - \omega)I - \omega R_2(\alpha)) \]
\[ J(\alpha) = -(R_1(\alpha) + R_2(\alpha)), \]
where \( R_1(\alpha) = \begin{pmatrix} 0 & 0 \\ -P_1(\alpha) & 0 \end{pmatrix} \) and \( R_2(\alpha) = \begin{pmatrix} 0 & -A_1(\alpha) \\ 0 & 0 \end{pmatrix} \).

Let \( \gamma \neq 0 \). It holds from Proposition 3.4 that
\[ \sigma(J(\alpha)) = \sigma(R(\gamma, \alpha)), \]
where \( R(\gamma, \alpha) = -\left(\frac{1}{\gamma}R_1 + \gamma R_2\right) \).
We assume that $\lambda \notin \sigma(L_\omega(\alpha))$. Since $(L_\omega(\alpha) - \lambda I)^{-1}$ is bounded, $\forall v \in H \times H$, the function $u$ defined by

$$u = \lambda^{\frac{1}{2}}\omega(L_\omega(\alpha) - \lambda I)^{-1}(I + \omega R_1(\alpha))^{-1}v,$$

(31)

satisfies

$$\|u\| \leq C\|v\|,$$

(32)

where $C$ is a constant independent of $v$.

Multiplying (31) by $(L_\omega(\alpha) - \lambda I)$ and substituting $L_\omega(\alpha)$ yields

$$(I + \omega R_1(\alpha))^{-1}((1 - \omega)I - \omega R_2(\alpha))u - \lambda u = \lambda^{\frac{1}{2}}\omega(I + \omega R_1(\alpha))^{-1}v,$$

which is equivalent to

$$(-\omega(R_2(\alpha) + \lambda R_1(\alpha)) - (\lambda + \omega - 1)I)u = \lambda^{\frac{1}{2}}\omega v.$$  

(33)

Multiplying (33) by $\lambda^{-\frac{1}{2}}\omega^{-1}$ yields

$$\left(R(\lambda^{-\frac{1}{2}}, \alpha) - \frac{(\lambda + \omega - 1)}{\lambda^{\frac{1}{2}}\omega} I\right) u = v.$$  

(34)

It follows that (34) admits a solution given by (31) which satisfies (32). Thus $\tau = \frac{(\lambda + \omega - 1)}{\lambda^{\frac{1}{2}}\omega} \notin \sigma(R(\lambda^{-\frac{1}{2}}, \alpha)) = \sigma(J(\alpha))$.

Conversely, assume that $\tau \notin \sigma(J(\alpha)) = \sigma(R(\lambda^{-\frac{1}{2}}, \alpha))$ and $\tau$ satisfies (30). $\forall v \in H \times H$, the function $u$ defined by

$$u = (\lambda^{\frac{1}{2}}\omega)^{-1}(R(\lambda^{-\frac{1}{2}}, \alpha) - \tau I)^{-1}(I + \omega R_1(\alpha))v,$$

(35)

satisfies the inequality (32). From (35), it follows that

$$(-\omega(R_2(\alpha) + \lambda R_1(\alpha)) - (\lambda + \omega - 1)I)u = (I + \omega R_1(\alpha))v,$$

or equivalently

$$((1 - \omega)I - \omega R_2(\alpha))u - \lambda(I + \omega R_1(\alpha))u = (I + \omega R_1(\alpha))v.$$  

(36)

Since $(I + \omega R_1(\alpha))$ is invertible, (36) is equivalent to

$$(I + \omega R_1(\alpha))^{-1}((1 - \omega)I - \omega R_2(\alpha))u - \lambda u = v.$$  

(37)

It follows that $\forall v \in H \times H$, the problem

$$(L_\omega(\alpha) - \lambda I)u = v$$

has a solution $u$ satisfying (32). Thus $\lambda \notin \sigma(L_\omega(\alpha))$. $\square$
Remark 3.6 The relation (30) between the spectrums of the operators of the Jacobi and SOR methods is the same as in the presence of finite dimensional linear systems with coefficient matrices possessing block Property A. Therefore, the results obtained in the last case may be generalized for the SOR acceleration of the PAS method.

In particular, we have theoretically that Jacobi and Gauss-Seidel methods converge simultaneously and the Gauss-Seidel method is two times faster than Jacobi method. We also have the following convergence results of the SOR method [7, 12, 16].

Proposition 3.7 Convergence of the SOR method.

1. If $\sigma(J(\alpha)) \subset \mathbb{R}$, the SOR method converges for $0 < \omega < 2$ and the optimal convergence parameter is

   $$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - (r(J(\alpha)))^2}},$$

   where $r(J(\alpha))$ is the spectral radius of $J(\alpha)$.

2. If $\sigma(J(\alpha))$ contains complex numbers, the SOR method converges if for some positive number $\tau \in (0, 1)$ and each $\lambda = \mu + i\beta \in \sigma(J(\alpha))$, the point $(\mu, \beta)$ lies in the interior of the ellipse $\epsilon(1, \tau) = \{(\mu, \beta) : \mu^2 + \beta^2 \tau^2 = 1\}$ and $0 < \omega < \frac{2}{1 + \tau}$. Theoretical results for the determination of the SOR optimal parameter are given in [16].

4 Numerical Results

We apply the PAS method on an example problem of particle transport.

Let $D = X \times W$, with $X = (0, 1)$ and $W = [-1, 1]$. We consider in $L^2(D)$ the following integro-differential equation:

$$\begin{cases}
\mu \frac{\partial u}{\partial x} + \sigma(x) u = \int_{-1}^{1} \kappa(r, \mu, \mu') u(r, \mu') d\mu' + q(r, \mu) \quad \text{in } D, \\
u(0, \mu) = 0, \mu > 0 \quad \text{and} \quad u(1, \mu) = 0, \mu < 0,
\end{cases}$$

where function $\sigma(x)$ and $\kappa(x, \mu, \mu')$ satisfy the following conditions:

1. $\sigma \in L^\infty((0, 1)), \exists \sigma_0 > 0$ such that $\sigma(x) \geq \sigma_0$ a.e. on $(0, 1)$;

2. $\kappa(r, \mu, \mu') > 0$ a.e. on $D$ and $\exists c \in (0, 1), \int_{-1}^{1} \kappa(r, \mu, \mu') d\mu' \leq \sigma_0 c$ a.e. on $W$. 
In the operator form, equation (39) reads:

\[ Tu = Au + Pu = q, \]

(40)

where the

\[ Au = \mu \frac{\partial u}{\partial x} \quad \text{and} \quad Pu = \sigma(x)u - \int_{1}^{1} \kappa(r, \mu, \mu') u(r, \mu') d\mu'. \]

(41)

The operator \( A \) is m-accretive in \( L^2(D) \) and it follows from the assumption made on \( \sigma \) and \( \kappa \) that operator \( P \) is positive definite [9]. The PAS iteration method can be applied for the solution of equation (39). The equation (39) is known to be near singular when \( c \approx 1 \) [9].

The discretization is carried out by a DSN scheme [9] consisting of using a finite set of \( L \) discrete angular directions \( \{\mu_k\}_{k=1}^{L} \in [-1, 1] \), which are nonzero and symmetric about the origin for the angular approximation and a difference method based on control volume approach and cell averaging for the spatial approximation.

For the numerical results, we took particular data for which an exact solution \( u \) is known: \( \sigma(x) = \sigma, \kappa(x, \mu, \mu') = \frac{3ac}{8} \mu^2, (0 < c < 1), \)

\[ q(x, \mu) = \begin{cases} 
\mu^2 + \sigma \mu x - \frac{3ac}{8} \mu^2, & \mu > 0 \\
\mu^2 + \sigma \mu (x - 1) - \frac{3ac}{8} \mu, & \mu < 0 
\end{cases} ; \quad u(x, \mu) = \begin{cases} 
\mu x, & \mu > 0 \\
\mu (x - 1), & \mu < 0 
\end{cases}. \]

We study the behavior of PAS, Jacobi (Jacobi_PAS), Gauss-Seidel (GS_PAS) and SOR (SOR_PAS) algorithms with respect to parameters \( \alpha, c \) and \( \sigma \). For iterative methods tested here, the iterations are stopped when the relative error \( \frac{\|U_{\text{exact}} - U^{(k)}\|}{\|U_{\text{exact}}\|} \) is less than a prescribed \( \epsilon > 0 \). The spatial mesh size is \( h = 1/30 \) and the angular mesh size is \( \tau = 1/500 \).

There is three sets of tests: one for fixed \( c \), another for fixed \( \sigma \) and the last for fixed \( \alpha \). For fixed \( \sigma \) or \( c \), we set \( \alpha = \sigma(1 - c) \). As shown by Figure 1 to Figure 4 all the methods converge. For \( \sigma = 100 \), we compare the \( c \)-dependence of the iterative methods used here (Figure 1 and Figure 2). We observe that Jacobi method is slower than PAS method which has the same convergence behavior as the Gauss-Seidel method except for the values of \( c > 0.98 \) (Figure 2), where the convergence of PAS seems to slow down. The \( \sigma \)-dependence of the iterative methods at fixed \( c = 0.8 \) is plotted by Figure 3 and Figure 4. We notice the convergence of all methods even for very large values of \( \sigma \). The Jacobi method remains slower than the PAS method which behaves as Gauss-Seidel method. It can be observed in the two sets of test that SOR is the fastest method. We also remark that, Gauss-Seidel is two times faster than Jacobi. This confirms the theoretical convergence results obtained. At fixed \( \alpha = 5 \), we compare the \( \sigma \)-dependence at \( c = 0.5 \) (Figure 5) and the \( c \)-dependence at \( \sigma = 100 \) (Figure 6) of the PAS and SOR methods. The SOR method gives excellent results than PAS. Figure 7 plots the convergence behavior of PAS and SOR methods at fixed \( \alpha = 10 \) for \( c \in \{0.5, 0.9\} \) and \( \sigma \in \{100, 1000\} \).
Comparison of the PAS, Jacobi, Gauss-Seidel and SOR (\(\omega = 0.95\)) methods at fixed \(\sigma = 100\), for \(c \in [0.1, 0.8]\) (\(\epsilon = 5E - 05\)).

Comparison of the PAS, Jacobi, Gauss-Seidel and SOR (\(\omega = 0.95\)) methods at fixed \(\sigma = 100\), for \(c \in [0.8, 0.99]\) (\(\epsilon = 5E - 04\)).

Comparison of the PAS, Jacobi, Gauss-Seidel and SOR (\(\omega = 0.95\)) methods at fixed \(c = 0.8\), for \(\sigma \in [1, 100]\) (\(\epsilon = 5E - 05\)).

Comparison of the PAS, Jacobi, Gauss-Seidel and SOR (\(\omega = 0.95\)) methods at fixed \(c = 0.8\), for large values of \(\sigma\) (\(\epsilon = 5E - 05\)).
Figure 5: Iteration number as function of $\sigma$ for PAS and SOR ($\omega = 0.95$) methods at fixed $c = 0.8$ and $\alpha = 5$ ($\epsilon = 5E - 04$).

Figure 6: Iteration number as function of $c$ for PAS and SOR ($\omega = 0.95$) methods at fixed $c\sigma = 100$ and $\alpha = 5$ ($\epsilon = 5E - 04$).

Figure 7: Convergence behavior of PAS and SOR ($\omega = 0.95$) methods at fixed $\alpha = 10$. 
5 Conclusion

Throughout this work, it comes that the two-step iterative method (PAS) for solving linear operator equations with operators admitting positive definite and $m$-accretive splitting in a Hilbert space $H$ converges. Moreover, analysis of a SOR acceleration of this method yields convergence results similar to those obtained in finite dimensional linear systems with coefficient matrices possessing Property A. Previous numerical results have shown the convergence of the PAS iteration and have demonstrated the effectiveness of the SOR acceleration of the PAS method.

References


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