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On a Subclass of Multivalent Analytic Functions with Negative Coefficients for Contraction Operators on Hilbert Space

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Abstract

In this paper the authors investigate a subclass $UCV(\alpha, \beta, \gamma, \delta, p; A)$ of p-valent analytic functions with negative coefficients and obtain some results concerning Characterisation Theorem, Coefficient Estimates and the Distortion Theorem. Also, we consider application of the fractional calculus for contraction operators on Hilbert Space.

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1 Introduction

Let $UCV(\alpha, \beta, \gamma, \delta, p;\ A)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p},\ (a_{n+p} \geq 0) \tag{1}$$

which are analytic and p-valent in the unit disk $E = \{z : |z| < 1\}$ and satisfy the condition

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \leq \beta\left|(\alpha+1)\frac{zf''(z)}{f'(z)} + (p-\gamma)\right| + (1-\delta)$$

$$0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ 0 \leq \delta \leq 1,\ 0 \leq \gamma < p,\ p \in N,\ |z| < 1$$

In particular, the class was studied by various authors [6,7,8,10].

Let $\mathcal{H}$ be a complex Hilbert Space. Let A be a contraction operator on $\mathcal{H}$ and $\sigma(A)$ denote its spectrum. For a complex function f which is analytic in the open neighbourhood $\mathcal{N}$ of $\sigma(A)$ in the complex plane, $f(A)$ will denote the operator on $\mathcal{H}$ defined by the usual Riesz-Dunford integral ([5], p. 568)

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - A)^{-1} dz$$

where I stands for the identity operator on $\mathcal{H}$, Γ is a positively oriented simple closed rectifiable contour such that the inside domain $\mathcal{D}$ of Γ contains $\sigma(A)$ and $\Gamma UD \subset \mathcal{N}$. The limit used in defining the integral is taken in the uniform topology of operators. $f(A)$ can also be defined by the series $f(A) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} A^n$ which converges in the norm topology [2].

Definition 1.1 *A function $f(z)$ of the form (1) is said to belong to the class $UCV(\alpha, \beta, \gamma, \delta, p; A)$ if $f(z)$ is analytic and p-valent in E and satisfy the condition*

$$\|Af''(A) - (p-1)f'(A)\| \leq \beta\|(\alpha+1)Af''(A) + (p-\gamma)f'(A)\| + (1-\delta)\|f'(A)\|$$

for $0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma < p,\ p \in N$ and all contraction A ($\|A\| < 1$) with $A \neq \Theta$ (Θ being the zero operator on $\mathcal{H}$).

If A^* denote the conjugate operator of A, then $\|A\| < 1$ will imply that $I > A^*A$

$$\text{i.e.,} \qquad ((I - A^*A)x, x) > 0.$$

In this paper we propose to prove a sufficient and necessary condition, coefficient estimate, distortion theorem for $UCV(\alpha, \beta, \delta, p; A)$ and to consider applications of the fractional calculus for operators. Such type of works were recently carried out by various authors [4,11,12,13].

2 MAIN RESULTS

Theorem 2.1 *The p-valent analytic function* $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ *is in the class* $UCV(\alpha, \beta, \gamma, \delta, p; A)$ *for all proper contraction* A *with* $\|A\| < 1$ *and* $A \neq \Theta$ *if and only if*

$$\sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}a_{n+p}$$

$$< p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\},\ n \geq 1 \qquad (2)$$

for $0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ 0 \leq \delta \leq 1,\ 0 \leq \gamma < p$ *and* $p \in N$. *The result is sharp for the function*

$$f(z) = z^p - \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}} z^{n+p}, n \geq 1.$$

Proof.

Assume that $f(z)$ of the form (1.1) is in the class $UCV(\alpha, \beta, \gamma, \delta, p; A)$. Then

$$\|Af''(A)-(p-1)\| < \beta\|(\alpha+1)Af''(A)+(p-\gamma)f'(A)\|+(1-\delta)\|f'(A)\|$$

gives

$$\left\|\sum_{n=1}^{\infty} n(n+p)a_{n+p}A^{n+p-1}\right\| \leq p\beta\left\|[(\alpha+1)(p-1)+(p-\gamma)]A^{p-1}\right.$$

$$\left.-\sum_{n=1}^{\infty}(n+p)[(\alpha+1)(n+p-1)+(p-\gamma)]a_{n+p}A^{n+p}\right\|+(1-\delta)\left\|pA^{p-1}-\sum_{n=1}^{\infty} a_{n+p}A^{n+p-1}\right\|$$

Choosing $A = eI$ $(0 < e < 1)$ and on simplification we get,

$$\sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}a_{n+p}e^{n+p-1}$$

$$< \{(1-\delta)p+p\beta[(\alpha+1)(p-1)+p-\gamma]e^{p-1}\}.$$

Letting $e \to 1$, we obtain

$$\sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}a_{n+p} < p\{\beta[(\alpha+1)(p-1)+(p-\gamma)+(1-\delta)]\}.$$

Conversely suppose the inequality (2) holds. Then

$$\|Af''(A)-(p-1)f'(A)\|-\beta\|(\alpha+1)Af''(A)+(p-\gamma)f'(A)\|+(1-\delta)\|f'(A)\|$$

$$\begin{aligned} = &\Big\|(n+p)a_{n+p-1}A^{n+p}\Big\| - p\beta\Big\|[(\alpha+1)(p-1)+(p-\gamma)]A^{p-1} \\ &- \sum_{n=1}^{\infty}(n+p)(n+p-1)(\alpha+1)+(p-\gamma)]a_{n+p}A^{n+p}\Big\| \\ &- (1-\delta)\Big\|pA^{p-1} - \sum_{n=1}^{\infty}(n+p)a_{n+p}A^{n+p-1}\Big\| \\ < &\sum_{n=1}^{\infty}(n+p)\Big\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\Big\}a_{n+p} \\ &- p\Big\{\beta[(\alpha+1)(p-1)+(p-\gamma)+(1-\delta)]\Big\} \leq 0 \end{aligned}$$

Hence $f(z) \in UCV(\alpha,\beta,\gamma,\delta,p;A)$.

Evidently, (2) is an extremal function for the class $UCV(\alpha,\beta,\gamma,\delta,p;A)$.

Corollary 2.2 *If* $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}z^{n+p}$ *belongs to the class* $UCV(\alpha,\beta,\gamma,\delta,p;A)$ *for* $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \delta \leq 1$, $0 \leq \gamma < p$, $p \in N$. $\|A\| < 1$, $A \neq \Theta$,*then*

$$a_{n+p} \leq \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{(n+p)\{(n+\delta-1+p\beta[(\alpha+1)(n+p-1)+(p-\gamma)]\}},\ n \geq 1.$$

This result is sharp for the function

$$f(z) = z^p - \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}}z^{n+p}, \quad n \geq 1.$$

Theorem 2.3 *Let* $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}z^{n+p}$ *belong to* $UCV(\alpha,\beta,\gamma,\delta,p;A)$ *for* $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \delta \leq 1$, $0 \leq \gamma < p$, $p \in N$, $\|A\| < 1$, $A \neq \Theta$. *Then*

$$\|A\|^p - \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{(1+p)\{\delta+p\beta[p(\alpha+1)+p-\gamma]\}}\|A\|^{p+1} \leq \|f(A)\|$$

$$\leq \|A\|^p + \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{(1+p)\{\delta+p\beta[p(\alpha+1)+p-\gamma]\}}\|A\|^{p+1}$$

and

$$p\|A\|^{p-1} - \frac{p^2\{\beta[(\alpha+1)(p-1)+p-\gamma]+1-\delta\}}{(1+p)\{\delta+p\beta[p(\alpha+1)+p-\gamma]\}}\|A\|^p \leq \|f'(A)\|$$

$$\leq p\|A\|^{p-1} + \frac{p^2\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta+p\beta[p(\alpha+1)+p-\gamma)\}}\|A\|^p.$$

The result is sharp for the function

$$f(z) = z^p - \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta+p\beta[(\alpha+1)p+p-\gamma]\}}z^{1+p}.$$

Proof.

$$(1+p)\{\delta+p\beta[(\alpha+1)p+p-\gamma]\}a_{n+p}$$
$$\leq (n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]+(1-\delta)\}, \forall n \geq 1.$$

Therefore, $(1+p)\{\delta+p\beta[(\alpha+1)p+p-\gamma]\}\sum_{n=1}^{\infty} a_{n+p}$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+(p-\gamma)]+(1-\delta)\}a_{n+p} \\ &\leq p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}, \end{aligned}$$

In view of Theorem 2.1.

$$\sum_{n=1}^{\infty} a_{n+p} \leq \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}$$

Therefore

$$\|f(A)\| \geq \|A\|^p - \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}\|A\|^{p+1}$$

and

$$\|f(A)\| \leq \|A\|^p + \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}\|A\|^{p+1}.$$

Note that

$$\frac{n+p}{p+1}(p+1)\{\delta+p\beta[(\alpha+1)(p-1)+(p-\gamma)]\}$$
$$\leq (n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}, n \geq 1.$$

$\sum_{n=1}^{\infty}(n+p)\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}a_{n+p}$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\} \\ &\leq p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}. \end{aligned}$$

That is

$$\sum_{n=1}^{\infty}(n+p)a_{n+p} \leq \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}.$$

Thus,

$$\begin{aligned} \|f'(A)\| &\geq p\|A\|^{p-1} - \|A\|^p\sum_{n=1}^{\infty}(n+p)a_{n+p} \\ &\geq p\|A\|^{p-1} - \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}\|A\|^p \end{aligned}$$

and

$$\|f'(A)\| \leq p\|A\|^{p-1} + \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{\{\delta+p\beta[(\alpha+1)(p-1)+p-\gamma]\}}\|A\|^{p} .$$

This completes the proof.

Theorem 2.4 *Let* $f_0(z) = z^p$,

$$f_n(z) = z^p - \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}}z^{n+p},\ n \geq 1.$$

Then a function $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}z^{n+p}$, $(a_{n+p} \geq 0)$ *is in the class* $UCV(\alpha,\beta,\gamma,\delta,p;A)$, *for* $0 \leq \alpha,\ \beta,\ \delta \leq 1,\ 0 \leq \gamma < p, p \in N,\ \|A\| < 1$, $A \neq \Theta$ *if and only if it can be written in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geq 0,\quad n = 0,1,2,\cdots$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof.

Suppose that $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}z^{n+p}$ is in the class $UCV(\alpha,\beta,\gamma,\delta,p;A)$. Then by Corollary 2.2

$$a_{n+p} \leq \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{(n+p)\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}} .$$

Set

$$\lambda_n = \frac{(n+p)\{(n+\delta-1)+p\beta[(\alpha-1)(n+p-1)+p-\gamma]\}}{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}},\quad n = 1,2,\ldots$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n \geq 0$. Then $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$.

Conversely suppose that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_n f_n(z), \quad \lambda_n \geq 0, \quad (n = 0,1,2,\ldots) \\ &= z^p - \sum_{n=1}^{\infty} \lambda_n \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}}z^{n+p}. \end{aligned}$$

Then we have,

$$\sum_{n=1}^{\infty} \frac{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)+(n+p-1)+(p-\gamma)]\}}{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}\lambda_n\mu_n ,$$

where

$$\mu_n = \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+p-\gamma]\}}$$

$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1$.

Thus in view of Theorem 2.1., $f(z) \in UCV(\alpha,\beta,\gamma,\delta,p;A)$

3 APPLICATIONS OF THE FRACTIONAL CALCULUS FOR OPERATORS OF ORDER k

Definition 3.1 *The fractional integral operator of order k is defined by*

$$D_A^{-k} f(A) = \frac{1}{\Gamma(k)} \int_0^1 A^k f(tA)(1-t)^{k-1} dt$$

where $k > 0$ and $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin.

Definition 3.2 *The fractional derivative for operator of order k is defined by*

$$D_A^k f(A) = \frac{1}{\Gamma(1-k)} g'(A), \text{ where } g(z) = \int_0^1 z^{1-k} f(tz)(1-t)^{-k} dt, \ (0 < k < 1)$$

and $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin.

Theorem 3.3 *If a function $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ belongs to the class $UCV(\alpha, \beta, \gamma, \delta, p; A)$ for $0 \le \alpha, \beta, \delta \le 1,\ \ 0 \le \gamma < p,\ p \in N$, then*

$$\|D_A^{-k} f(A)\| \ge \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \|A\|^{p+k} - \sigma(\alpha, \beta, \gamma, \delta, p)$$

and

$$\|D_A^{-k} f(A)\| \le \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \|A\|^{p+k} + \sigma(\alpha, \beta, \gamma, \delta, p),$$

where

$$\sigma(\alpha, \beta, \gamma, \delta, p) = \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}}{(1+p)\{\delta + p\beta[(\alpha+1)p+(p-\gamma)]\}} \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \|A\|^{p+k+1}$$

for $k > 0$ and all invertible operators A with $(A^{\frac{1}{q}})^* A^{\frac{1}{q}} = A^{\frac{1}{q}} (A^{\frac{1}{q}})^*$, $(q \in N)$, $\|A\| < 1$ and $\rho(A)\rho(A^{-1}) \le 1$, where $\rho(A)$ is the spectral radius of A.

Proof. Consider the function

$$\begin{aligned} R(A) &= \frac{\Gamma(p+k+1)}{\Gamma(p+1)} A^{-k} D_A^{-k} f(A) \\ &= A^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)}{\Gamma(n+p+k+1)} \frac{\Gamma(p+k+1)}{\Gamma(p+1)} a_{n+p} A^{n+p} \\ &= A^p - \sum_{n=1}^{\infty} t_{n+p} A^{n+p}, \text{ where } t_{n+p} = \frac{\Gamma(n+p+1)}{\Gamma(n+p+k+1)} \frac{\Gamma(p+k+1)}{\Gamma(p+1)} \end{aligned}$$

and note that $0 < t_{n+p} < 1$.

Thus we obtain

$\sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+(p-\gamma)]\}t_{n+p}$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty}(n+p)\{(n+\delta-1)+p\beta[(\alpha+1)(n+p-1)+(p-\gamma)]\}a_{n+p} \\ &\leq p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}, \quad \text{since } 0 < t_{n+p} < 1. \end{aligned}$$

Therefore the function $R(z) \in UCV(\alpha, \beta, \gamma, \delta, p; A)$.

Hence by Theorem 2.3, we have

$$\|D_A^{-k} f(A)\| \leq \frac{\Gamma(p+1)}{\Gamma(p+k+1)}\|A\|^k\|A\|^p + \sigma(\alpha, \beta, \gamma, \delta, p)$$

and

$$\|D_A^{-k} f(A)\| \geq \frac{\Gamma(p+1)}{\Gamma(p+k+1)}\|A\|^k\|A\|^p - \sigma(\alpha, \beta, \gamma, \delta, p),$$

where

$$\sigma(\alpha, \beta, \gamma, \delta, p) = \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]\}\Gamma(p+1)}{(1+p)\{\delta+p\beta[(\alpha+1)p+(p-\gamma)]\Gamma(p+k+1)\}}\|A\|^{p+k+1}$$

since $(A^{\frac{1}{q}})^*(A^{\frac{1}{q}}) = (A^{\frac{1}{q}})(A^{\frac{1}{q}})^*$, $q \in N$ and Corollary 3.8 in [3].

Theorem 3.4 *Let* $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ *belong to the class* $UCV(\alpha, \beta, \gamma, \delta, p; A)$ *for* $0 \leq \alpha,\ \beta, \delta \leq 1$, $0 \leq \gamma < p$, $p \in N$. *Then*

$$\|D_A^k f(A)\| \geq \frac{\Gamma(p+1)}{\Gamma(p+1-k)}\|A\|^{p-k} - \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}\Gamma(p+1)}{\{\delta+p\beta[(\alpha+1)p+(p-\gamma)]\Gamma(p+1-k)\}}\|A\|^{p+1-k}$$

and

$$\begin{aligned} \|D_A^k f(A)\| \leq{} & \frac{\Gamma(p+1)}{\Gamma(p+1+k)}\|A\|^{p-k} \\ & + \frac{p\{\beta[(\alpha+1)(p-1)+p-\gamma]+(1-\delta)\}\Gamma(p+1)}{\{\delta+p\beta[(\alpha+1)(p-1)(p-\gamma)]\}\Gamma(p+1-k)}\|A\|^{p+1-k} \end{aligned}$$

for $0 < k < 1$ *and all invertible operators* A *with*

$$(A^{\frac{1}{q}})^*(A^{\frac{1}{q}}) = (A^{\frac{1}{q}})(A^{\frac{1}{q}})^*, \ (q \in N), \ \|A\| < 1$$

and $\rho(A)\rho(A^{-1}) \leq 1$, *where* $\rho(A)$ *is the spectral radius of* A.

Proof.

Consider the function

$$\begin{aligned} T(A) &= \frac{\Gamma(p+1-k)}{\Gamma(p+1)}A^k D_A^k f(A) \\ &= A^p - \sum_{n=1}^{\infty} a_{n+p}\frac{\Gamma(p+1-k)}{\Gamma(p+1)}\frac{\Gamma(n+p-1)}{\Gamma(n+p+1-k)}A^{n+p}. \end{aligned}$$

Then by Theorem 2.3,

$$\begin{aligned}\|T(A)\| &\geq \|A\|^p - \|A\|^{p+1}\sum_{n=1}^{\infty}(n+p)a_{n+p} \\ &\geq \|A\|^p - \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{\{\delta+p\beta[(\alpha+1)(p-1)+(p-\gamma)]\}}\,\|A\|^{p+1}\end{aligned}$$

and

$$\|T(A)\| \leq \|A\|^p + \frac{p\{\beta[(\alpha+1)(p-1)+(p-\gamma)]+(1-\delta)\}}{\{\delta+p\beta[(\alpha+1)(p-1)+(p-\gamma)]\}}\,\|A\|^{p+1}.$$

Note that, if $S_{n+p} = \frac{\Gamma(p+1-k)\Gamma(n+p-1)}{\Gamma(p+1)\Gamma(n+p+1-k)}$ then $0 < s_{n+p} < (n+p)$.

Following the lines of argument for Theorem 3.1, we get the desired estimates.

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