A-Poisson Structures on Weil Bundles

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Abstract

Let $M$ be a paracompact differentiable manifold, $A$ a Weil algebra and $M^A$ a manifold of infinitely near points on $M$ of kind $A$. We define the notion of $A$-Poisson manifold on $M^A$. We show that when $M$ is a Poisson manifold, then $M^A$ is an $A$-Poisson manifold. We also show that if $(M, \Omega)$ is a symplectic manifold, the structure of $A$-Poisson manifold on $M^A$ defined by $\Omega^A$ coincide with the prolongation on $M^A$ of the Poisson structure on $M$ defined by the symplectic form $\Omega$.

Mathematics Subject Classification: 17D63, 53D17, 53D05, 58A32

Keywords: Weil bundle, Weil algebra, Poisson manifold, symplectic manifold, $A$-Poisson manifold

1 Introduction

In what follows, $M$ denotes a paracompact differentiable manifold, $C^\infty(M)$ the algebra of smooth functions on $M$, $A$ a Weil algebra i.e a real commutative algebra with unit, of finite dimension, and with an unique maximal ideal $m$ of codimension 1 over $\mathbb{R}$. In this case, there exists an integer $h$ such that $m^{h+1} = (0)$ and $m^h \neq (0)$. The integer $h$ is the height of $A$. Also we have $A = \mathbb{R} \oplus m$.

For example the algebra of dual numbers $\mathbb{D} = \mathbb{R}[T]/(T^2)$ is a Weil algebra with height 1.

We recall that a near point of $x \in M$ of kind $A$ is a morphism of algebras

$$\xi : C^\infty(M) \rightarrow A$$

such that $[\xi(f) - f(x)] \in m$ for any $f \in C^\infty(M)$. 
We denote $M^A_x$ the set of near points of $x \in M$ of kind $A$ and

$$M^A = \bigcup_{x \in M} M^A_x$$


We have $\mathbb{R}^A = A$ and $M^D = TM$ is the tangent bundle of $M$.

When the dimension of $M$ is $n$, then the dimension of $M^A$ is $n \times \dim(A)$ [11]. Let $(U, \varphi)$ be a local chart with local coordinates $(x_1, x_2, ..., x_n)$. The application

$$U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), \xi(x_2), ..., \xi(x_n)),$$

is a bijection from $U^A$ to an open of $A^n$. Thus $M^A$ is an $A$-manifold of dimension $n$.

The set, $C^\infty(M^A, A)$, of smooth functions on $M^A$ with values in $A$ is a commutative algebra with unit over $A$.

For any $f \in C^\infty(M)$, the application

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f),$$

is smooth and the application

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A,$$

is a monomorphism of algebras.

The following assertions are equivalent [1]:

1. $X$ is a derivation of $C^\infty(M^A)$ i.e. $X$ is a vector field on $M^A$;

2. $X : C^\infty(M) \longrightarrow C^\infty(M^A, A)$ is a $\mathbb{R}$-linear application such that, for any $f, g \in C^\infty(M)$,

$$X(fg) = X(f) \cdot g^A + f^A \cdot X(g)$$

i.e. $X$ is a derivation from $C^\infty(M)$ to $C^\infty(M^A, A)$ with respect the module structure

$$C^\infty(M^A, A) \times C^\infty(M) \longrightarrow C^\infty(M^A, A), (F, f) \longmapsto F \cdot f^A.$$ 

Thus the set, $\mathfrak{X}(M^A)$, of vector fields on $M^A$ considered as derivations of $C^\infty(M)$ into $C^\infty(M^A, A)$ is a module over $C^\infty(M^A, A)$.

When

$$\theta : C^\infty(M) \longrightarrow C^\infty(M)$$
is a vector field on $M$, then the application
\[ \theta^A : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto [\theta(f)]^A, \]
is a vector field on $M^A$. We say that the vector field $\theta^A$ is the prolongation to $M^A$ of the vector field $\theta$.

If $X$ is a vector field on $M^A$, considered as a derivation of $C^\infty(M)$ into $C^\infty(M^A, A)$, then there exists, [1], an unique derivation
\[ \widetilde{X} : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A) \]
such that

1. $\widetilde{X}$ is $A$-linear;
2. $\widetilde{X} [C^\infty(M^A)] \subset C^\infty(M^A)$;
3. $\widetilde{X}(f^A) = X(f)$ for any $f \in C^\infty(M)$.

Let $(a_\alpha)_{\alpha = 1, \ldots, r}$ be a basis of $A$ and $(a^*_\alpha)_{\alpha = 1, \ldots, r}$ be the dual basis.

If
\[ Y : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A) \]
is an $A$-linear derivation such that
\[ Y(f^A) = \widetilde{X}(f^A) \]
for any $f \in C^\infty(M)$, then
\[ Y [C^\infty(M^A)] \subset C^\infty(M^A) \]
since
\[ Y(a^*_\alpha \circ f^A) = \widetilde{X}(a^*_\alpha \circ f^A) \in C^\infty(M^A) \]
for any $\alpha = 1, 2, \ldots, r$. Thus, [1], $Y = \widetilde{X}$.

The application
\[ [,] : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto \widetilde{X} \circ Y - \widetilde{Y} \circ X, \]
is $A$-bilinear and defines a structure of $A$-Lie algebra on $\mathfrak{X}(M^A)$ [1].

If we denote $\text{Der} [C^\infty(M^A, A)]$, the $C^\infty(M^A, A)$-module of derivations of $C^\infty(M^A, A)$, then the application
\[ \mathfrak{X}(M^A) \longrightarrow \text{Der} [C^\infty(M^A, A)], X \longmapsto \widetilde{X}, \]
is a morphism of $A$-Lie algebras [1].

For any $p \in \mathbb{N}$,
\[ \Lambda^p(M^A, A) = \mathcal{L}^p_{sk} \left[ \mathfrak{X}(M^A), C^\infty(M^A, A) \right] \]
denotes the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree $p$ on $\mathfrak{X}(M^A)$. We say that $\Lambda^p(M^A, A)$ is the $C^\infty(M^A, A)$-module of differential $A$-forms of degree $p$ on $M^A$. We have
\[ \Lambda^0(M^A, A) = C^\infty(M^A, A). \]

We denote
\[ \Lambda(M^A, A) = \bigoplus_{p=0}^n \Lambda^p(M^A, A). \]

If $\omega$ is a differential form of degree $p$ on $M$, then there exists an unique differential $A$-form of degree $p$ on $M^A$ such that
\[ \omega^A(\theta^A_1, \theta^A_2, ..., \theta^A_p) = [\omega(\theta_1, \theta_2, ..., \theta_p)]^A \]
for any vector fields $\theta_1, \theta_2, ..., \theta_p \in \mathfrak{X}(M)$. We say that the differential $A$-form $\omega^A$ is the prolongation to $M^A$ of the differential form $\omega$ [4], [7].

When
\[ d : \Lambda(M) \rightarrow \Lambda(M) \]
is the exterior differentiation operator, we denote
\[ d^A : \Lambda(M^A, A) \rightarrow \Lambda(M^A, A) \]
the cohomology operator associated to the representation
\[ \mathfrak{X}(M^A) \rightarrow \text{Der}\left[ C^\infty(M^A, A) \right], X \mapsto \tilde{X}. \]

We recall that for $\eta \in \Lambda^p(M^A, A)$, we have
\[
(d^A \eta)(X_1, X_2, ..., X_{p+1})
= \sum_{i=1}^{p+1} (-1)^{i-1} \tilde{X}_i \left[ \eta(X_1, X_2, ..., \hat{X}_i, ..., X_{p+1}) \right]
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1})
\]
for any vector fields $X_1, X_2, ..., X_{p+1}$ on $M^A$, where $\hat{X}_i$ means that the term $X_i$ is omitted.
The application
\[ d^A : \Lambda(M^A, A) \longrightarrow \Lambda(M^A, A) \]
is \(A\)-linear and
\[ d^A(\omega^A) = (d\omega)^A \]
for any \(\omega \in \Lambda(M)\) \([1]\). It is obvious that if
\[ d\omega = 0, \]
then
\[ d^A(\omega^A) = 0. \]

Let \((M, \omega)\) be a symplectic manifold. Then the manifold \(M\) is a Poisson manifold i.e. the algebra \(C^\infty(M)\) carries a structure of Poisson algebra. For any linear form
\[ \psi : A \longrightarrow \mathbb{R}, \]
the differential form \(\psi \circ \omega^A\) is not necessary a symplectic form on \(M^A\). That means that the prolongation \(\omega^A\) does not always induce a structure of Poisson on \(M^A\). In effect, let \(m\) be the maximal ideal of a local algebra \(A\),
\[ \text{ann}(m) = \{ a \in A / a \cdot x = 0 \text{ for any } x \in m \} \]
and
\[ \mu_A : A \times A \longrightarrow A, (a, b) \longmapsto a \cdot b, \]
the multiplication on \(A\). Then there exists a linear form \(\psi : A \longrightarrow \mathbb{R}\) such that the bilinear symmetric form
\[ \psi \circ \mu_A : A \times A \longrightarrow \mathbb{R} \]
is nondegenerated if and only if \(\dim [\text{ann}(m)] = 1\) \([7]\).

When \((M, \omega)\) is a symplectic manifold and \(\psi \in A^*\) a linear form on \(A\), then the scalar 2-form \(\psi \circ \omega^A\) is a symplectic form on \(M^A\) if and only if \(\dim [\text{ann}(m)] = 1\) and \(\psi [\text{ann}(m)] \neq 0\) : it is the case when the local algebra is \(A = \mathbb{R}[T_1, ..., T_s] / [T_1^{k_1}, ..., T_s^{k_s}]\). Thus, when \((M, \omega)\) is a symplectic manifold, we cannot obtain a Poisson structure on \(M^A\) which comes from the prolongation of \(\omega\) when \(\dim [\text{ann}(m)] \neq 1\). For example, it is the case when \(A = \mathbb{R}[T_1, T_2] / (T_1, T_2)^2\).
In this paper, we do not study the structures of $A$-manifolds but we study the structures on $M^A$ as an $A$-manifold. When $M$ is a manifold, the basic algebra of $M$ is $C^\infty(M)$. As $\mathfrak{X}(M^A)$ is a $C^\infty(M^A, A)$-module, considered as the set of derivations of $C^\infty(M)$ to $C^\infty(M^A, A)$, and a Lie algebra over $A$, and as $M^A$ is an $A$-manifold, that means that the basic algebra of $M^A$ is $C^\infty(M^A, A)$: thus the natural space for studying Poisson structures on $M^A$ is $C^\infty(M^A, A)$ but not $C^\infty(M^A)$. When $(M, \omega)$ is a symplectic manifold, we will show that $(M^A, \omega^A)$ is a symplectic $A$-manifold.

The main goal of this paper is to define the notion of $A$-Poisson structures on $M^A$ and to show that if a manifold $M$ is a Poisson manifold, then $M^A$ admits an $A$-Poisson structure. We also show that if $M$ is a symplectic manifold, then $M^A$ admits an $A$-Poisson structure such that this structure coincide with the structure of $A$-Poisson manifold on $M^A$ deduced by the structure of Poisson manifold on $M$ defined by the symplectic form.

2 $A$-Poisson structures

We recall that a Poisson structure on a differentiable manifold $M$ is due to the existence of a bracket $\{,\}$ on $C^\infty(M)$ such that the pair $(C^\infty(M), \{,\})$ is a real Lie algebra and

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

for any $f, g, h \in C^\infty(M)$. In this case we say that $M$ is a Poisson manifold and $C^\infty(M)$ is a Poisson algebra.

We will say that the $A$-algebra $C^\infty(M^A, A)$ is a Poisson $A$-algebra if there exists a bracket $\{,\}$ on $C^\infty(M^A, A)$ such that the pair $(C^\infty(M^A, A), \{,\})$ is a Lie algebra over $A$ satisfying

$$\{\varphi, \psi_1 \cdot \psi_2\} = \{\varphi, \psi_1\} \cdot \psi_2 + \psi_1 \cdot \{\varphi, \psi_2\}$$

for any $\varphi, \psi_1, \psi_2 \in C^\infty(M^A, A)$. When $C^\infty(M^A, A)$ is a Poisson $A$-algebra, we will say that the manifold $M^A$ is a $A$-Poisson manifold or $M^A$ admits an $A$-Poisson structure.

2.1 Structure of $A$-Poisson manifold on $M^A$ when $M$ is a Poisson manifold

In this part, $M$ is a Poisson manifold with bracket $\{,\}$. In this case, for any $f \in C^\infty(M)$, the application

$$ad(f) : C^\infty(M) \rightarrow C^\infty(M), g \mapsto \{f, g\},$$

is a vector field on $M$ and, for any $g \in C^\infty(M)$, we get
\[
\text{ad}(fg) = f \cdot \text{ad}(g) + g \cdot \text{ad}(f).
\]

For any $f \in C^\infty(M)$, let
\[
[\text{ad}(f)]^A : C^\infty(M) \to C^\infty(M^A, A), g \mapsto \{f, g\}^A,
\]
be the prolongation of the vector field $\text{ad}(f)$ and let
\[
[\text{ad}(f)]^A : C^\infty(M^A, A) \to C^\infty(M^A, A)
\]
be the unique $A$-linear derivation such that
\[
[\text{ad}(f)]^A(g^A) = [\text{ad}(f)]^A(g) = \{f, g\}^A
\]
for any $g \in C^\infty(M)^1$.

**Proposition 1** For any $\varphi \in C^\infty(M^A, A)$, the application
\[
\tau_\varphi : C^\infty(M) \to C^\infty(M^A, A), f \mapsto -[\text{ad}(f)]^A(\varphi),
\]
is a vector field on $M^A$.

**Proof.** It is obvious that $\tau_\varphi$ is linear. For any $f, g \in C^\infty(M)$, we have
\[
\tau_\varphi(fg) = -[\text{ad}(fg)]^A(\varphi)
\]
\[
= -[f \cdot \text{ad}(g) + g \cdot \text{ad}(f)]^A(\varphi)
\]
\[
= f^A \cdot \left(-[\text{ad}(g)]^A(\varphi)\right) + g^A \cdot \left(-[\text{ad}(f)]^A(\varphi)\right)
\]
\[
= \left(-[\text{ad}(f)]^A\right)(\varphi) \cdot g^A + f^A \cdot \left(-[\text{ad}(f)]^A\right)(\varphi)
\]
\[
= \tau_\varphi(f) \cdot g^A + f^A \cdot \tau_\varphi(g).
\]
That ends the proof. ■

For any $\varphi \in C^\infty(M^A, A)$, we denote
\[
\widetilde{\tau}_\varphi : C^\infty(M^A, A) \to C^\infty(M^A, A)
\]
the unique $A$-linear derivation such that
\[
\widetilde{\tau}_\varphi(f^A) = \tau_\varphi(f)
\]
for any $f \in C^\infty(M)$.

For $f \in C^\infty(M)$, we verify that
\[
\widetilde{\tau}_{f^A} = [\text{ad}(f)]^A.
\]
Proposition 2 For \( \varphi, \psi \in C^\infty(M^A, A) \) and for \( a \in A \), we have
\[
\tilde{\tau}_{\varphi + \psi} = \tilde{\tau}_{\varphi} + \tilde{\tau}_{\psi}; \\
\tilde{\tau}_{a \varphi} = a \cdot \tilde{\tau}_{\varphi}; \\
\tilde{\tau}_{\varphi \psi} = \varphi \cdot \tilde{\tau}_{\psi} + \psi \cdot \tilde{\tau}_{\varphi}.
\]

Proof. For \( \varphi, \psi \in C^\infty(M^A, A) \), \( \tilde{\tau}_{\varphi} + \tilde{\tau}_{\psi} \) is an \( A \)-linear derivation. For any \( f \in C^\infty(M) \), we get
\[
(\tilde{\tau}_{\varphi} + \tilde{\tau}_{\psi})(f^A) = (\tilde{\tau}_{\varphi})(f^A) + (\tilde{\tau}_{\psi})(f^A)
\]
\[
= \left(-[\text{ad}(f)]^A\right)(\varphi) + \left(-[\text{ad}(f)]^A\right)(\psi)
\]
\[
= \left(-[\text{ad}(f)]^A\right)(\varphi + \psi)
\]
\[
= (\tilde{\tau}_{\varphi + \psi})(f^A).
\]
We deduce that
\[
\tilde{\tau}_{\varphi + \psi} = \tilde{\tau}_{\varphi} + \tilde{\tau}_{\psi}.
\]

For \( \varphi \in C^\infty(M^A, A) \), \( a \in A \) and \( f \in C^\infty(M) \), we have
\[
(a \cdot \tilde{\tau}_{\varphi})(f^A) = a \cdot (\tilde{\tau}_{\varphi})(f^A)
\]
\[
= a \cdot \left(-[\text{ad}(f)]^A\right)(\varphi)
\]
\[
= \left(-[\text{ad}(f)]^A\right)(a \cdot \varphi)
\]
\[
= (\tilde{\tau}_{a \varphi})(f^A).
\]
We deduce that
\[
\tilde{\tau}_{a \varphi} = a \cdot \tilde{\tau}_{\varphi}.
\]

For \( \varphi, \psi \in C^\infty(M^A, A) \), \( \varphi \cdot \tilde{\tau}_{\psi} + \psi \cdot \tilde{\tau}_{\varphi} \) is an \( A \)-linear derivation. For any \( f \in C^\infty(M) \), we get
\[
[\varphi \cdot \tilde{\tau}_{\psi} + \psi \cdot \tilde{\tau}_{\varphi}](f^A) = \varphi \cdot \tilde{\tau}_{\psi}(f^A) + \psi \cdot \tilde{\tau}_{\varphi}(f^A)
\]
\[
= \varphi \cdot \left(-[\text{ad}(f)]^A\right)(\psi) + \psi \cdot \left(-[\text{ad}(f)]^A\right)(\varphi)
\]
\[
= \left(-[\text{ad}(f)]^A\right)(\varphi \cdot \psi)
\]
\[
= \tilde{\tau}_{\varphi \psi}(f^A).
\]
We deduce that
\[
\tilde{\tau}_{\varphi \psi} = \varphi \cdot \tilde{\tau}_{\psi} + \psi \cdot \tilde{\tau}_{\varphi}.
\]
That ends the proof. ■

For any $\varphi, \psi \in C^\infty(M^A, A)$, we let

$$\{\varphi, \psi\}_A = \widetilde{\tau}_\varphi(\psi).$$

In what follows, we will show that this bracket defines a structure of Poisson $A$-algebra on $C^\infty(M^A, A)$.

**Proposition 3** The application

$$\{\cdot, \cdot\}_A : C^\infty(M^A, A) \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), (\varphi, \psi) \longmapsto \{\varphi, \psi\}_A,$$

is $A$-bilinear and skew-symmetric.

**Proof.** It is obvious that this application is $A$-bilinear. For any $\varphi \in C^\infty(M^A, A)$, we verify that the application

$$H_\varphi : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), \psi \longmapsto \widetilde{\tau}_\varphi(\psi) + \widetilde{\tau}_\psi(\varphi)$$

is an $A$-linear derivation. The application

$$\sigma_\varphi : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto \widetilde{\tau}_\varphi(f^A) + \widetilde{\tau}_f(\varphi),$$

is a vector field on $M^A$ considered as a derivation of $C^\infty(M)$ into $C^\infty(M^A, A)$. As for $f \in C^\infty(M)$, we have

$$H_\varphi(f^A) = \widetilde{\tau}_\varphi(f^A) + \widetilde{\tau}(\varphi) = (\sigma_\varphi)(f^A).$$

We deduce, [1], that

$$H_\varphi = \sigma_\varphi.$$

On the other hand, we have

$$(\sigma_\varphi)(f^A) = (\sigma_\varphi)(f)$$

$$= \widetilde{\tau}_\varphi(f^A) + \widetilde{\tau}_f(\varphi)$$

$$= \left(-[\text{ad}(f)]^A\right)(\varphi) + \left([\text{ad}(f)]^A\right)(\varphi)$$

$$= 0$$

for any $f \in C^\infty(M)$. Thus we conclude that $\sigma_\varphi = 0$ i.e. $H_\varphi = 0$. For any $\psi \in C^\infty(M^A, A)$, we get

$$H_\varphi(\psi) = 0$$
\[ \tilde{\tau}_\varphi(\psi) + \tilde{\tau}_\psi(\varphi) = 0. \]

Thus
\[ \{ \varphi, \psi \}_A = -\{ \psi, \varphi \}_A . \]

As the characteristic is different of 2, therefore
\[ \{ \varphi, \varphi \}_A = 0 \]
for any \( \varphi \in C^\infty(M^A, A) \).

\[ \text{Proposition 4} \]
For any \( \varphi, \psi_1, \psi_2 \in C^\infty(M^A, A) \), then
\[ \{ \varphi, \psi_1 \cdot \psi_2 \}_A = \{ \varphi, \psi_1 \}_A \cdot \psi_2 + \psi_1 \cdot \{ \varphi, \psi_2 \}_A . \]

\[ \text{Proof.} \]
For any \( \varphi \in C^\infty(M^A, A) \), as
\[ \tilde{\tau}_\varphi : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A) \]
is an \( A \)-linear derivation, then we have
\[ \{ \varphi, \psi_1 \cdot \psi_2 \}_A = \tilde{\tau}_\varphi(\psi_1 \cdot \psi_2) = \tilde{\tau}_\varphi(\psi_1) \cdot \psi_2 + \psi_1 \cdot \tilde{\tau}_\varphi(\psi_2) = \{ \varphi, \psi_1 \}_A \cdot \psi_2 + \psi_1 \cdot \{ \varphi, \psi_2 \}_A . \]

That ends the proof.

\[ \text{Proposition 5} \]
For \( f \in C^\infty(M) \) and \( \varphi \in C^\infty(M^A, A) \), the application
\[ H_{(f^A, \varphi)} : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A) , \]
defined by
\[ H_{(f^A, \varphi)}(\psi) = \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(\psi) \right] - \tilde{\tau}_\psi \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(\psi) \right] \]
for any \( \psi \in C^\infty(M^A, A) \), is an \( A \)-linear derivation which is zero.

\[ \text{Proof.} \]
In fact for \( a \in A \), we have
\[ H_{(f^A, \varphi)}(a \cdot \psi) \]
\[ = \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(a \cdot \psi) \right] - \tilde{\tau}_a \psi \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(a \cdot \psi) \right] \]
\[ = \tilde{\tau}_\varphi \left[ a \cdot \tilde{\tau}_{f^A}(\psi) \right] - (a \cdot \tilde{\tau}_\psi) \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ a \cdot \tilde{\tau}_\varphi(\psi) \right] \]
\[ = a \cdot \left( \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(\psi) \right] - \tilde{\tau}_\psi \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(\psi) \right] \right) \]
\[ = a \cdot H_{(f^A, \varphi)}(\psi). \]
We also have, for $\psi_1, \psi_2 \in C^\infty(M^A, A)$,

$$H_{(f^A, \varphi)}(\psi_1 + \psi_2)$$

$$= \tilde{\varphi} \left[ \tilde{f}^A(\psi_1 + \psi_2) \right] - \tilde{\psi}_{1+\psi_2} \left[ \tilde{f}^A(\varphi) \right] - \tilde{f}^A \left[ \tilde{\varphi}(\psi_1 + \psi_2) \right]$$

$$= \tilde{\varphi} \left[ \tilde{f}^A(\psi_1) + \tilde{f}^A(\psi_2) \right] - \tilde{\psi}_{\psi_1} \left[ \tilde{f}^A(\varphi) \right] - \tilde{f}^A \left[ \tilde{\varphi}(\psi_1) \right]$$

Thus the application $H_{(f^A, \varphi)}$ is an $A$-linear derivation.

The application

$$\sigma_{(f^A, \varphi)} : C^\infty(M) \longrightarrow C^\infty(M^A, A),$$

defined by

$$\sigma_{(f^A, \varphi)}(g) = \tilde{\varphi} \left[ \tilde{f}^A(g^A) \right] - \tilde{\psi}_g^A \left[ \tilde{f}^A(\varphi) \right] - \tilde{f}^A \left[ \tilde{\varphi}(g^A) \right]$$

for any $g \in C^\infty(M)$, is a vector field on $M^A$. It is obvious that $\sigma_{(f^A, \varphi)}$ is linear.
For $g, h \in C^\infty(M)$, we get

$$
\sigma_{(f^A, \varphi)}(gh) = \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(gh)^A \right] - \tilde{\tau}_{(gh)^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(gh)^A \right]
= \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(g^A \cdot h^A) \right] - \tilde{\tau}_{g^A \cdot h^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(g^A \cdot h^A) \right]
= \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(g^A) \cdot h^A + g^A \cdot \tilde{\tau}_{f^A}(h^A)^A \right] - g^A \cdot \tilde{\tau}_{h^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right]
= \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(\varphi) \right] \cdot h^A + \tilde{\tau}_{f^A}(g^A) \cdot h^A - \tilde{\tau}_{f^A}(h^A) - h^A \cdot \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(h^A) \right]
= \tilde{\tau}_\varphi \left( \tilde{\tau}_{f^A}(g^A) \right) \cdot h^A - g^A \cdot \tilde{\tau}_{h^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right] - h^A \cdot \tilde{\tau}_{g^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right]
= \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(g^A) \right] \cdot h^A + g^A \cdot \tilde{\tau}_{f^A}(h^A) - \tilde{\tau}_{f^A}(g^A) \cdot \tilde{\tau}_\varphi(h^A)
= \sigma_{(f^A, \varphi)}(g) \cdot h^A + g^A \cdot \sigma_{(f^A, \varphi)}(h).
$$

The application $\sigma_{(f^A, \varphi)}$ is a vector field on $M^A$.

It is obvious that

$$
H_{(f^A, \varphi)}(g^A) = \sigma_{(f^A, \varphi)}(g)
$$

for any $g \in C^\infty(M)$. Thus, [1], we have

$$
H_{(f^A, \varphi)} = \tilde{\sigma}_{(f^A, \varphi)}.
$$

On the other hand,

$$
\tilde{\sigma}_{(f^A, \varphi)}(g^A) = \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(g^A) \right] - \tilde{\tau}_{g^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right] - \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(g^A) \right]
= \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(g^A) \right] - \tilde{\tau}_{g^A} \left[ \tilde{\tau}_{f^A}(\varphi) \right] + \tilde{\tau}_{f^A} \left[ \tilde{\tau}_\varphi(g^A) \right]
= \tilde{\tau}_\varphi \left[ \tilde{\tau}_{f^A}(g^A) \right] + \left[ \tilde{\tau}_{f^A}, \tilde{\tau}_{g^A} \right] (\varphi)
= \tilde{\tau}_\varphi \left( \{f, g\}^A \right) + [ad(\{f, g\})]^A(\varphi)
= -[ad(\{f, g\})]^A(\varphi) + [ad(\{f, g\})]^A(\varphi)
= 0
$$

for any $g \in C^\infty(M)$. We have, [1],

$$
\tilde{\sigma}_{(f^A, \varphi)} = 0
$$

i.e. $H_{(f^A, \varphi)} = 0$. ■
Proposition 6 For any $\varphi, \psi \in C^\infty(M^A, A)$, then

$$[\widetilde{\tau}_\varphi, \widetilde{\tau}_\psi] = \widetilde{\tau}_{\{\varphi, \psi\}_A}.$$

Proof. For $f \in C^\infty(M)$ and $\varphi \in C^\infty(M^A, A)$, as $H(f^A, \varphi) = 0$, then

$$H(f^A, \varphi)(\psi) = 0$$

for any $\psi \in C^\infty(M^A, A)$. Thus, we obtain

$$\tilde{\tau}_{f^A} [\tilde{\tau}_\varphi(\psi)] = \tilde{\tau}_\varphi [\tilde{\tau}_{f^A}(\psi)] - \tilde{\tau}_\psi [\tilde{\tau}_{f^A}(\varphi)]$$

$$\{ f^A, \{ \varphi, \psi \}_A \}_A = \{ \varphi, \{ f^A, \psi \}_A \}_A - \{ \psi, \{ f^A, \varphi \}_A \}_A$$

$$- \{ \{ \varphi, \psi \}_A, f^A \}_A = - \{ \varphi, \{ \psi, f^A \}_A \}_A + \{ \psi, \{ \varphi, f^A \}_A \}_A.$$  

i.e.

$$\{ \{ \varphi, \psi \}_A, f^A \}_A = \{ \varphi, \{ f^A, \psi \}_A \}_A - \{ \psi, \{ f^A, \varphi \}_A \}_A$$

$$\tilde{\tau}_{\{\varphi, \psi\}_A}(f^A) = (\tilde{\tau}_\varphi \circ \tilde{\tau}_\psi)(f^A) - (\tilde{\tau}_\psi \circ \tilde{\tau}_\varphi)(f^A)$$

$$= [\tilde{\tau}_\varphi, \tilde{\tau}_\psi](f^A).$$

As for any $f \in C^\infty(M)$, we have

$$[\tilde{\tau}_\varphi, \tilde{\tau}_\psi] (f^A) = \tilde{\tau}_{\{\varphi, \psi\}_A}(f^A).$$

Therefore, [1],

$$[\tilde{\tau}_\varphi, \tilde{\tau}_\psi] = \tilde{\tau}_{\{\varphi, \psi\}_A}.$$  

That ends the proof. $\blacksquare$

We now will show the identity of Jacobi.

Proposition 7 For any $\varphi, \psi, \phi \in C^\infty(M^A, A)$, then

$$\{ \varphi, \{ \psi, \phi \}_A \}_A + \{ \psi, \{ \phi, \varphi \}_A \}_A + \{ \phi, \{ \varphi, \psi \}_A \}_A = 0.$$

Proof. For $\varphi, \psi, \phi \in C^\infty(M^A, A)$, we obtain

$$\{ \varphi, \{ \psi, \phi \}_A \}_A + \{ \psi, \{ \phi, \varphi \}_A \}_A + \{ \phi, \{ \varphi, \psi \}_A \}_A$$

$$= \{ \varphi, \{ \psi, \phi \}_A \}_A - \{ \psi, \{ \phi, \varphi \}_A \}_A - \{ \{ \varphi, \psi \}_A, \phi \}_A$$

$$= \tilde{\tau}_\varphi [\tilde{\tau}_\psi(\phi)] - \tilde{\tau}_\psi [\tilde{\tau}_\varphi(\phi)] - \tilde{\tau}_{\{\varphi, \psi\}_A}(\phi)$$

$$= ([\tilde{\tau}_\varphi, \tilde{\tau}_\psi] - \tilde{\tau}_{\{\varphi, \psi\}_A})(\phi)$$

$$= 0.$$

That ends the proof. $\blacksquare$

Thus, we have shown the following theorem:
Theorem 8 If $M$ is a Poisson manifold with bracket $\{,\}$, then $M^A$ is an $A$-Poisson manifold with the bracket

$$\{,\}_A : C^\infty(M^A, A) \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), (\varphi, \psi) \longmapsto \{\varphi, \psi\}_A = \tilde{\tau}_\varphi(\psi).$$

In this case, we will say that the structure of $A$-Poisson manifold on $M^A$ defined by $\{,\}_A$ is the prolongation on $M^A$ of the structure of Poisson manifold on $M$ defined by $\{,\}$.

2.2 Structure of $A$-Poisson manifold on $M^A$ when $M$ is a symplectic manifold

Proposition 9 If $\omega$ is a differential form on $M$ and if $\theta$ is a vector field on $M$, then

$$(i_\theta \omega)^A = i_\theta^A(\omega^A).$$

Proof. If the degree of $\omega$ is $p$, then $(i_\theta \omega)^A$ is the unique differential $A$-form of degree $p-1$ such that

$$(i_\theta \omega)^A(\theta^A_1, ..., \theta^A_{p-1}) = [(i_\theta \omega)(\theta_1, ..., \theta_{p-1})]^A = [\omega(\theta, \theta_1, ..., \theta_{p-1})]^A$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$. As $i_\theta^A(\omega^A)$ is of degree $p-1$ and is such that

$$i_\theta^A(\omega^A)[\theta^A_1, ..., \theta^A_{p-1}] = \omega^A(\theta^A_1, \theta^A_1, ..., \theta^A_{p-1}) = [\omega(\theta, \theta_1, ..., \theta_{p-1})]^A$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$, we conclude that $(i_\theta \omega)^A = i_\theta^A(\omega^A)$. ■

Proposition 10 If $(M, \Omega)$ is a symplectic manifold, then the application

$$\mathfrak{X}(M^A) \longrightarrow \Lambda^1(M^A, A), X \longmapsto i_X \Omega^A,$$

is an isomorphism of $C^\infty(M^A, A)$-modules.

Proof. Let $X$ be a vector field on $M^A$ such that $i_X \Omega^A = 0$. Let $\xi \in M^A$ with origin $x_0 \in M$. As $\Omega$ is a symplectic form, we can choose a system of local coordinates $(x_1, x_2, ..., x_{2n})$ on an open $U$, $x_0 \in U$, such that

$$\Omega|_U = \sum_{i=1}^n dx_i \Lambda dx_{i+n}.$$
Thus $\xi \in U^A$ and
\[
\Omega^A|_{U^A} = \sum_{i=1}^{n} d^A(x^A_i) \Lambda d^A(x^A_{i+n}).
\]

As $i_X \Omega^A = 0$, by writing $X|_{U^A} = \sum_{i=1}^{n} f_i \left( \frac{\partial}{\partial x_i} \right)^A + \sum_{i=1}^{n} f_{i+n} \left( \frac{\partial}{\partial x_{i+n}} \right)^A$ where $f_i, f_{i+n} \in C^\infty(U^A, A)$ for $i = 1, 2, ..., n$, we have
\[
0 = [i_X|_{U^A} \Omega^A|_{U^A}] \left( \left( \frac{\partial}{\partial x_i} \right)^A \right)
\]
\[
= -f_{i+n}
\]
and
\[
0 = [i_X|_{U^A} \Omega^A|_{U^A}] \left( \left( \frac{\partial}{\partial x_{i+n}} \right)^A \right)
\]
\[
= f_i
\]
for $i = 1, 2, ..., n$. Thus $X|_{U^A} = 0$. Therefore $X(\xi) = 0$. As $\xi$ is arbitrary, we have $X = 0$. The application
\[
\mathfrak{X}(M^A) \longrightarrow \Lambda^1(M^A, A), X \mapsto i_X \Omega^A,
\]
is injective.

Let $\eta \in \Lambda^1(M^A, A)$, $\xi \in M^A$ with origin $x_0 \in U$ where $(U, \varphi)$ is a chart with local coordinates $(x_1, x_2, ..., x_{2n})$ such that
\[
\Omega|_U = \sum_{i=1}^{n} dx_i \Lambda dx_{i+n}
\]
and
\[
\Omega^A|_{U^A} = \sum_{i=1}^{n} d^A(x^A_i) \Lambda d^A(x^A_{i+n}).
\]

By writing
\[
\eta|_{U^A} = \sum_{i=1}^{n} h_i d^A(x^A_i) + \sum_{i=1}^{n} h_{i+n} d^A(x^A_{i+n}),
\]
where $h_i, h_{i+n} \in C^\infty(U^A, A)$ for $i = 1, 2, ..., n$, we verify that the vector field
\[
\theta|_{U^A} = \sum_{i=1}^{n} h_{i+n} \left( \frac{\partial}{\partial x_i} \right)^A - \sum_{i=1}^{n} h_i \left( \frac{\partial}{\partial x_{i+n}} \right)^A
\]
is such that \( i_{\theta_{U^A}}\Omega^A|_{U^A} = \eta|_{U^A} \). If \((V, \psi)\) is an other chart around \(x_0\) with local coordinates \((x'_1, x'_2, ..., x'_{2n})\) such that

\[
\Omega|_V = \sum_{i=1}^{n} dx'_i \Lambda dx'_{i+n}
\]

and

\[
\Omega^A|_{V^A} = \sum_{i=1}^{n} d^A(x'_i^A) \Lambda d^A(x'_{i+n}^A).
\]

We have

\[
\eta|_{U^A \cap V^A} = (\eta|_{U^A})|_{U^A \cap V^A} = (i_{\theta_{U^A}}|_{U^A \cap V^A} \Omega^A)|_{U^A \cap V^A}
\]

and

\[
\eta|_{U^A \cap V^A} = (\eta|_{V^A})|_{U^A \cap V^A} = (i_{\theta_{V^A}}|_{U^A \cap V^A} \Omega^A)|_{U^A \cap V^A}
\]

Thus \( \theta_{U^A}|_{U^A \cap V^A} = \theta_{V^A}|_{U^A \cap V^A} \). If \((U_i)_{i \in I}\) is a covering of \(M\) with such opens, then there exists a vector field \(X\) on \(M^A\) such that

\[
X|_{U_i^A} = \theta_{U^A_i}.
\]

We have \( \eta = i_X \Omega^A \) and we conclude that the application

\[
\mathfrak{X}(M^A) \longrightarrow \Lambda^1(M^A, A), X \longmapsto i_X \Omega^A,
\]

is surjective. ■

**Corollary 11** When \((M, \Omega)\) is a symplectic manifold, then \((M^A, \Omega^A)\) is a symplectic \(A\)-manifold.

When \((M, \Omega)\) is a symplectic manifold, for any \(f \in C^\infty(M)\), we denote \(X_f\) the unique vector field on \(M\) such that

\[
i_{X_f} \Omega = df
\]

and for any \(\varphi \in C^\infty(M^A, A)\), we denote \(X_\varphi\) the unique vector field on \(M^A\), considered as a derivation of \(C^\infty(M)\) into \(C^\infty(M^A, A)\), such that

\[
i_{X_\varphi} \Omega^A = d^A(\varphi).
\]
In this case, we know that
\[ X_f = ad(f). \]

We easily verify that the bracket
\[
\{ \varphi, \psi \}_\Omega^A = -\Omega^A(X_\varphi, X_\psi) = \widetilde{X}_\varphi(\psi)
\]
defines a structure of $A$-Poisson manifold on $M^A$.

**Proposition 12** If $(M, \Omega)$ is a symplectic manifold, for any $f \in C^\infty(M)$ then
\[
X_f^A = (X_f)^A.
\]

**Proof.** The differential $A$-form
\[
i_{(X_f)^A} \Omega^A
\]
is the unique differential $A$-form of degree 1 such that
\[
\left[ i_{(X_f)^A} \Omega^A \right](\theta^A) = \Omega^A((X_f)^A, \theta^A) = \left[ \Omega(X_f, \theta) \right]^A
\]
for any $\theta \in \mathfrak{X}(M)$. On the other hand, the differential $A$-form $i_{X_f} \Omega^A$ is of degree 1 and is such that
\[
\left[ i_{X_f} \Omega^A \right](\theta^A) = \Omega^A(X_f^A, \theta^A) = \widetilde{\theta}^A(f^A) = \theta^A(f) = [\theta(f)]^A = [(df) (\theta)]^A = (df)^A (\theta^A) = [i_{X_f} \Omega]^A (\theta^A) = [\Omega(X_f, \theta)]^A
\]
for any $\theta \in \mathfrak{X}(M)$. We conclude that
\[
i_{(X_f)^A} \Omega^A = i_{X_f^A} \Omega^A.
\]

Thus, we deduce that $X_f^A = (X_f)^A$. ■

We state the following theorem:
**Theorem 13** If $(M, \Omega)$ is a symplectic manifold, the structure of $A$-Poisson manifold on $M^A$ defined by $\Omega^A$ coincide with the prolongation on $M^A$ of the Poisson structure on $M$ defined by the symplectic form $\Omega$.

**Proof.** We will show that

$$\tilde{\tau}_\varphi = \tilde{X}_\varphi$$

for any $\varphi \in C^\infty(M^A, A)$. For any $f \in C^\infty(M)$, we have

$$\tilde{X}_\varphi(f^A) = \left[ d^A (f^A) \right] (X_\varphi)$$

$$= \left[ i_{X_f^A} \Omega^A \right] (X_\varphi)$$

$$= -\Omega^A (X_\varphi, X_f^A)$$

$$= -\Omega^A [X_\varphi, (X_f)^A]$$

$$= -\left[ i_{X_\varphi} \Omega^A \right] ((X_f)^A)$$

$$= -(d^A \varphi)((X_f)^A)$$

$$= -(X_f)^A(\varphi)$$

$$= -[ad (f)]^A(\varphi)$$

$$= \tilde{\tau}_\varphi(f^A).$$

We deduce, [1], that

$$\tilde{\tau}_\varphi = \tilde{X}_\varphi.$$

Therefore, for any $\varphi, \psi \in C^\infty(M^A, A)$, we have

$$\{\varphi, \psi\}_{\Omega^A} = \{\varphi, \psi\}_A.$$

That ends the proof. ■

**References**


Received: June, 2011