Krein's Method and Volterra – Fredholm Integral Equation

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Abstract

Here, the existence of a unique solution of Volterra – Fredholm integral equation (V-FIE) of the first kind is considered in the space $L_2 [-1,1] \times C (0,T),$ $T \leq 1.$ The Fredholm integral term is considered in position with discontinuous kernel, while the Volterra integral term is considered in time with continuous kernel. Using a numerical, we have system of Fredholm integral equations (SFIEs) of the first find. Using Krein's method, the solution of SFIEs is obtained in the form of spectral relationships (SRs). Finally, many special cases and applications in fluid mechanics and contact problems are discussed.

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Introduction

Singular integral equations of the first kind have been received considerable interest in the mathematical literatures, because of their many field of applications in different areas of sciences, for example see [1- 4]. The solution of these IEs can be obtained analytically using one of the following methods: Cauchy method [5], potential theory method [6], orthogonal polynomials method [7], integral transformation methods [4-7] and Krein's method [8]. Mkhitarian and Abdou, in [9] discussed some different methods for solving the FIE of the first kind with logarithmic kernel.

In this work, we consider the V-FIE of the first kind

\[
\int_{a}^{a} F(t, \tau) \, k \left( \frac{x-y}{\lambda} \right) \phi(y, \tau) \, dy \, d\tau = \pi f(x, t) = \pi \left[ r(t) - f(x) \right]
\]

\[
((x, y) \in [-1, 1], (t, \tau) \in [0, T], T \leq 1; \lambda \in (0, \infty))
\]

(1)

\[
k(z) = \int_{0}^{\infty} \frac{L(v) \cos v z}{v} \, dv, \quad L(v) = \frac{m + v}{1 + v}, \quad m \geq 1,
\]

(2)

under the condition

\[
\int_{a}^{a} \phi(x, t) \, dx = P(t)
\]

(3)

The function \( L(v) \) is continuous and positive for \( v \in (0, \infty) \) and satisfies the following asymptotic equalities

\[
L(v) = m - (m-1)v + O\left( v^3 \right), \quad v \to 0
\]

\[
L(v) = 1 - \frac{m-1}{v} + O\left( \frac{1}{v^3} \right) \quad (v \to \infty, m \geq 1)
\]

(4)
The V-FIE (1), under the condition (2), can be investigated from the contact problem of a rigid surface \((G, \nu)\) having an elastic material occupying the domain \([-a, a]\), where \(f_s(x)\) is describing the surface base of a stamp. This stamp is impressed into an elastic layer surface by a variable known force \(P(t)\), \(t \in [0, T]\), \(T \leq 1\), whose eccentricity of application \(e(t)\), that case a rigid displacement \(\gamma(t)\). Here, \(G\) is called the displacement magnitude and \(\nu\) is Poisson's coefficient.

In order to guarantee the existence of unique solution of (1), we assume, for the two constants \(E\) and \(D\), the following conditions:

(i) The kernel of position satisfies

\[
\int_a^{-a} \int_a^{-a} k^2 \left( \frac{x - y}{\lambda} \right) dx dy \leq E,
\]

(ii) The positive continuous kernel, which represents the resistance force of the material, \(F(t, \tau) \in C([0, T] \times [0, T])\) and satisfies \(F(t, \tau) < D\),

(iii) The continuous function of time \(\gamma(t) \in C[0, T]\), while the position function \(f_s(x) \in L_2[-a, a]\) and \(f(x, t) \in L_2[-a, a] \times C[0, T]\). The norm of \(f(x, t)\) is

\[
\|f\|_{L_2 \times C} = \max_{t} \left( \int_a^{-a} f^2(x, \tau) dx \right)^{1/2} d\tau.
\]

(iv) The unknown potential function \(\phi(x, t)\) satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

In this work, we use a numerical method to transform the V-FIE into linear SFIEs of the first kind. Then, using Krein’s method, the solution of SFIEs can be obtained in the form of spectral relationships (SRs) of CPs. Many special cases are derived and discussed from the work. Moreover, some applications in contact problems and fluid mechanics are considered.
2. System of FIEs

If we divide the interval \([0,T]\), \(0 \leq t \leq T \leq 1\) as \(0 \leq t_0 < t_1 < \cdots < t_N = T\), when \(t=t_k\), \(k=0, 1, 2, \ldots, \ell\). The V-FIE (1) takes the form, see [2].

\[
\int_{0}^{t} \int_{a}^{a} F(t, \tau) k(x, y) \phi(y, \tau) \, dy \, d\tau = \sum_{j=0}^{k} u_j F_{j,k} \int_{a}^{a} k(x, y) \varphi_j(y) \, dy = \pi f_k(x)
\]

(5)

In (5) we neglect the error term, \(O(h_i^{p+1})\) where \(h_i = \max h_j\), \(h_j = t_{j+1} - t_j\). The constant \(u_j\) defined as the characteristic number, see [2]. Also we used the following notations

\[
\phi(x, t_i) = \phi_i(x), \quad F(t_i, t_j) = F_{i,j}, \quad f(x, t_i) = f_i(x).
\]

(6)

The boundary condition (3), becomes

\[
\int_{a}^{a} \phi_k(x) \, dx = P_k \quad (P_k \text{ are constants}),
\]

(7)

Let, in (5), \(m=1\) and \(\lambda \to \infty\), such that the term \((x-y)\) is very small. Then, using the relation [7]

\[
\int_{0}^{\infty} \frac{\cos v \, \zeta}{v} \, dv = -\ln \zeta + d \quad (d \text{ is a constant}),
\]

(8)

the formula (5) becomes

\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{a}^{a} \ln \left| \frac{1}{x-y} \right| \phi_j(y) \, dy = \pi g_k(x),
\]

(9)

\[
g_k(x) = f_k(x) - \frac{P_k}{\pi} \sum_{j=0}^{\ell} u_j F_{j,\ell}
\]

The formula (9) represents linear SFIEs of the first kind with logarithmic kernel.

To obtain the solution, we use Krein’s method, see [8,9].
3. Krein’s method see [8,9]

In this section, the solution of the linear SFIEs with logarithmic kernel, using Krein’s method is considered. Applying the principle formulas of Krein’s method, see [8, 9], the corresponding solution of (9) under the condition (7) is given as

\[
 u_j F_{j,\ell} \phi_j^\ell (x) = \frac{1}{\pi} \left[ \frac{1}{\ln 2 + d} \right] \cdot \frac{1}{\sqrt{a^2 - x^2}} \\
 - \frac{2}{\pi^2} \int_a^x \frac{du}{\sqrt{u^2 - x^2}} \frac{d}{du} \left[ u \frac{d}{du} \int_0^u \frac{g_j^\ell (y)dy}{\sqrt{u^2 - y^2}} \right] \\
\]

(10)

where

\[
 u_j F_{j,\ell} \phi_j^\ell (x) = -\frac{2}{\pi^2} \frac{d}{dx} \int_a^x \frac{du}{\sqrt{u^2 - x^2}} \int_0^u \frac{d g_j^\ell (y)}{\sqrt{u^2 - y^2}}, \\
 I(u) = \frac{2}{\pi} \left[ \int_0^u \frac{g_j^\ell (y)dy}{\sqrt{u^2 - y^2}} + u \ln \left( \frac{2}{u + d} \right) \frac{d}{du} \int_0^u \frac{g_j^\ell (y)dy}{\sqrt{u^2 - y^2}} \right] \\
\]

(11)

(12)

and

\[
 g_j^\ell (x) = g_j^\ell (x) + g_j^\ell (x), \quad \phi_j^\ell (x) = \phi_j^\ell (x) + \phi_j^\ell (x) \\
 \phi_j^\ell (-x) = \pm \phi_j^\ell (x), \quad g_j^\ell (-x) = \pm g_j^\ell (x) \quad (x \in (-a, a), \quad j = 1, 2, \ldots, \ell, \quad \ell = 1, 2, \ldots, N) \\
\]

(13)

4. Method of solution

To obtain the solution we state the following theorem

**Theorem 1:** The SRs for the SFIEs (9), under (7), take the form
\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{-\ln(\frac{x-y}{a})+d}{\sqrt{a^2-y^2}} T_{n_j} \left( \frac{x}{a} \right) dy = \begin{cases} \\
\pi P_{\ell} \left[ \ln \left( \frac{z}{a} \right) + d \right] & n_\ell = 0 \\
\pi T_{n_\ell} \left( \frac{x}{a} \right), \ell = 1, \ldots, N & n_\ell \geq 1
\end{cases}
\]

where

\[ T_{n_j}(x), j = 1, 2, \ldots, \ell \text{ are the CPs of the first kind and order } n. \]

**Proof:** The proof of (14) depends on the following lemmas

**Lemma 1:** For all positive integers \(n_j, a = 1\), we have

\[
I_{n_j}(u) = 2 \left[ \frac{1}{2} P_{n_j}^{(-1,0)} \left( 2u^2 - 1 \right) + n_j u^2 \ln \left( \frac{2}{u} + d \right) P_{n_{j-1}}^{(0,1)} \left( 2u^2 - 1 \right) \right]
\]

where \( P_{n_j}^{(\alpha,\beta)}(x) \) are Jacobi polynomials

**Proof:** To prove (15) let \( g_\ell^+(y) = T_{2n_\ell}(y) \), where \( T_{2n_\ell}(y) \) are the CPs of the first kind, then (15) can be written in the form

\[
I_{n_j}(u) = \frac{2}{\pi} \left[ D_{n_j}(u) + u \ln \left( \frac{2}{u} + d \right) \frac{d}{du} D_{n_\ell}(u) \right]
\]

where

\[
D_{n_j}(u) = \int_{0}^{u} \frac{T_{2n_j}(s) ds}{\sqrt{u^2 - s^2}}.
\]

Using the substitution \( s = ut \) and the relation \( T_{2n_j}(x) = T_{n_j}(2x^2 - 1) \), the formula (17) takes the form

\[
D_{n_j} = \int_{0}^{1} \left( 1 - t^2 \right) \frac{1}{2} T_{n_j} \left( 2t^2 - 1 \right) dt.
\]

Using the famous relation between CPs, LPs and JPS, see [10]

\[
\int_{-1}^{1} \left( 1 - t^2 \right)^{\frac{1}{2}} T_{n_j} \left( 1 - t^2 y \right) dt = \frac{\pi}{2} \left[ P_{n_j}(1 - y) + P_{n_{j-1}}(1 - y) \right],
\]
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\[
2 P_n^{(-1,0)}(x) = P_n(x) - P_{n-1}(x),
\]

where \( P_n(x) \) are Legendre polynomials, the formula (18), yields

\[
D_n(u) = \frac{\pi}{2} P_n^{(-1,0)} \left( 2u^2 - 1 \right)
\]

(20)

Also, the first derivative of (20) takes the form

\[
\frac{d}{du} D_n(u) = n \pi u P_n^{(0,1)} \left( 2u^2 - 1 \right) \quad n = 1, 2, \ldots; \ell = 1, 2, \ldots, N
\]

(21)

( \( P_n^{(\alpha,\beta)}(x) = 0 \) for negative integer ).

Finally, introducing (20), (21) in (16), we obtain the required result.

**Corollary 1:** Put \( u = 1 \) in (15), we have

\[
D_n(1) = 2 \left[ \frac{1}{2} P_n^{(-1,0)}(1) + n \pi \ln(2 + d) P_n^{(0,1)}(1) \right]
\]

(22)

Using the famous relation \( P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(1 + \alpha)} \), the formula (22) becomes

\[
D_n(1) = 2n \pi \ln(2 + d)
\]

(23)

where \( \Gamma(x) \) is the Gamma function.

**Corollary 2:** The value \( \frac{d}{du} \left( u \frac{dD_n}{du} \right) \) is given by

\[
\frac{d}{du} \left( u \frac{dD_n}{du} \right) = D_n^{(2)}(u) = 2n \pi u \left[ P_n^{(0,1)}(2u^2 - 1) + (n + 1)u^2 P_n^{(1,2)}(2u^2 - 1) \right].
\]

(24)
Lemma 2: The value of the following integral

\[ A_{n_\ell}(x) = \frac{1}{x} \frac{du}{\sqrt{u^2 - x^2}} \int \frac{d^2}{du^2} \left( \int_{0}^{T_{2n_\ell}} (s) ds \right) \]

takes the form

\[ A_{n_\ell}(x) = \frac{\pi^2 n_\ell!}{\sqrt{2\Gamma(n_\ell - \frac{1}{2})}} \frac{1}{\sqrt{1 - y}} \left[ \frac{1 - y}{2n_\ell - 1} P^{(\frac{1}{2}, \frac{1}{2})}_{n_\ell - 1}(y) - (1 + y) P^{(\frac{1}{2}, \frac{1}{2})}_{n_\ell - 1}(y) \right] \]

\[ + \frac{2n_\ell \pi}{\sqrt{1 - y}} \]

(\(y = 2x^2 - 1\), \(n_\ell = 1,2,\ldots; \ell = 1,2,\ldots, N\)).

Proof: For proving the lemma, we introduce (25) in (26) to have

\[ A_{n_\ell}(x) = 2n_\ell \pi \left[ \frac{1}{x} \int \frac{uP^{(0,1)}_{n_\ell - 1}(2u^2 - 1) du}{\sqrt{u^2 - x^2}} + (n_\ell + 1) \int \frac{u^3 P^{(1,2)}_{n_\ell - 2}(2u^2 - 1) du}{\sqrt{u^2 - x^2}} \right] \]

Assume in (27) the substitution \(2u^2 - 1 = y, \ 2x^2 - 1 = z\), to have

\[ A_{n_\ell}(x) = A_{n_\ell}(z) = \frac{\pi n_\ell}{\sqrt{2}} \left[ \frac{1}{z} \int \frac{P^{(0,1)}_{n_\ell - 1}(y) dy}{\sqrt{y - z}} + n_\ell (n_\ell + 1) \pi \frac{1}{2\sqrt{2}} \int \frac{yP^{(1,2)}_{n_\ell - 2}(y) dy}{\sqrt{y - z}} \right] \]

\[ + \frac{n_\ell (n_\ell + 1) \pi}{2\sqrt{2}} \int \frac{P^{(1,2)}_{n_\ell - 2}(y) dy}{\sqrt{y - z}} \]

If we put \(y = 1 - (1 - z)v\), then (28) yields

\[ A_{n_\ell}(z) = \frac{\pi n_\ell}{\sqrt{2}} \int_{0}^{1} \left( 1 - (1 - z)^v \right)^{\frac{1}{2}} P^{(0,1)}_{n_\ell - 1}(1 - (1 - z)^v) dv \]
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\[
+ \frac{m_\ell(n_\ell + 1)}{2\sqrt{2}} \sqrt{1 - z(1 + z)} \int_0^1 (1 - v)^{-1} P_{n_\ell - 2}^{(1, 2)} [1 - (1 - z)v] dv
\]

\[
+ \frac{n_\ell(n_\ell + 1)\pi}{2\sqrt{2}} (1 - z)^{3/2} \int_0^1 (1 - v)^{1/2} P_{n_\ell - 2}^{(1, 2)} [1 - (1 - z)v] dv.
\]

(29)

If we use the famous formulas [10]

\[
\int_0^1 z^{\lambda - 1} (1 - z)^{r - 1} P_n^{(\alpha, \beta)}(1 - z) dz =
\]

\[
\frac{\Gamma(\alpha + n + 1)\Gamma(\lambda)\Gamma(r)}{n!\Gamma(1 + \alpha)\Gamma(\lambda + r)} {}_2F_1\left(-n, n + \alpha + \beta + 1; \lambda, \alpha + 1, \lambda + r; \frac{\gamma}{2}\right),
\]

\[
(R_\ell \lambda > 0, R_\ell r > 0),
\]

(30)

and

\[
P_n^{(\alpha, \beta)}(v) = \binom{n + \alpha}{n} {}_2F_1\left(-n, n + \alpha + \beta + 1; 1 - \frac{v}{2}\right)
\]

(31)

where \( {}_2F_1\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z\right) \) is the generalized hypergeometric series and \( F(\alpha, \beta; \gamma; z) \) is the hypergeometric Gauss function, the first integral term of (29) becomes

\[
\int_0^1 (1 - v)^{1/2} P_{n_\ell - 1}^{(0, 1)} [1 - (1 - z)v] dv = \frac{\sqrt{\pi(n_\ell - 1)!}}{\Gamma(n_\ell + 1/2)} P_{n_\ell - 1}^{(1/2)}(z)
\]

(32)

Also, using the same way, the second and third integral term of (29), yield

\[
\int_0^1 (1 - v)^{-1/2} P_{n_\ell - 2}^{(1, 2)} [1 - (1 - z)v] dv =
\]

\[
\frac{1}{1 - z} \left\{ -2\sqrt{\pi} \Gamma(n_\ell) \frac{(n_\ell + 1)\Gamma(n_\ell - 1/2)}{(n_\ell + 1/2)\Gamma(n_\ell - 1/2)} P_{n_\ell - 1}^{(3/2)}(z) + \frac{2}{n_\ell + 1} \right\}
\]

(33)

And
\[
\int_0^1 (1-v)^{1/2} P^{(1,2)}_{n-2} [1-(1-z)v] dv =
\]
\[
-\sqrt{\pi} (n-1)! \over (n+1)(1-z) \Gamma(n+1) + {1 \over 2} \right] P^{1/2 \pm 1/2}_{n-1} (z) + {2 \over (n+1)(1-z)}
\]

(34)

Introducing the three formulas (32) - (34) in (23) the lemma is proved.

Finally, to prove the theorem, we write (23) in the \textbf{CPs} form, for this purpose, we must consider the following famous formulas, see [10,11]

\begin{enumerate}
\item \[
P_n^{(-1/2,1/2)}(2x^2-1) = \frac{\Gamma(n+1/2)\Gamma(\lambda)}{\sqrt{\pi} \Gamma(n+\lambda)} C^{\lambda}_{2n}(x)
\]

(35)

(\text{Relation between Jacobi and Gegenbauer polynomials})

\item \[
P_n^{(1/2,1/2)}(2x^2-1) = \frac{\Gamma(n+3/2)\Gamma(\lambda)}{\sqrt{\pi} \Gamma(n+\lambda+1)} C^{\lambda}_{2n+1}(x)
\]

(36)

and

\item \[
\lim_{\lambda \to 0} \Gamma(\lambda) C^{\lambda}_{n}(x) = \frac{2}{n} T_n(x) \quad (n \geq 1)
\]

(37)

(\text{Relation between Chebyshev and Gegenbauer polynomials})
\end{enumerate}

Using these famous relations in (28), one has

\[
A_{n,1} (x) = \frac{n \pi [1-T_{2n} (x)]}{\sqrt{1-x^2}} \quad , \quad (n \geq 1)
\]

(38)

Introducing (38) and (23) in (11), the theorem is proved.

By using the same way, we can prove this theorem

\textbf{Theorem 2}: The \textbf{SRs} for the \textbf{SFIEs} with the kernel defined by (9) and the known function is odd is given by
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\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \left[ \frac{1}{\ln |x-s|} + d \right] T_{2n_j-1}(s) ds = \frac{\pi}{2n_{\ell} - 1} T_{2n_{\ell}-1}(x)
\]

\[(n_{\ell} \geq 1, \ell = 1, 2, \ldots, N) \quad (39)\]

5. Conclusion and results

From the above results and discussion, the following may be concluded

(1) The contact problem of a rigid surface of an elastic material, when a stamp of length \(2a\) is impressed into an elastic layer surface of a strip by a variable \(P(t), 0 \leq t \leq T \leq 1\), whose eccentricity of application \(e(t)\), represents a \textbf{V-FIE} of the first kind.

(2) The \textbf{V-FIE} can be transformed into \textbf{SFIEs}, using a numerical method.

(3) The \textbf{SFIEs} depends on the number of derivatives of \(F(t, \tau)\) with respect to time \(t, t \in [0, T] \) \(T \leq 1\).

(4) The displacement problems of ant plane deformation of an infinite rigid strip with width \(2a\), putting on an elastic layer of thickness \(h\) is considered as a special case of this work when \(t = 1, F(t, \tau) = 1, f(x, t) = H\) and \(\varphi(x, 1) = \psi(x)\). Here \(H\) represents the displacement magnitude and \(\psi(x)\) the unknown function represents the displacement stress.

(5) The problems of infinite rigid strip with width \(2a\) impressed in a viscous liquid layer of thickness \(h\), when the strip has a velocity resulting from the impulsive force \(v = v_0 e^{-iwt} , i = \sqrt{-1}\), where \(v_0\) is the constant velocity, \(w\) is the angular velocity resulting rotating the strip about z-axis are considered as special case of this work, when \(F(t, \tau) = \text{ constant}\, ,\, \text{and } t = 1\) see [4].

(6) In the discussion (4) and (5), when \(h \to \infty\), this means that the depth of the liquid (Fluid mechanics) or the thickness of elastic material (contact problem) becomes an infinite.
(7) The three kinds of the displacement problem, in the theory of elasticity and mixed contact problems, which discussed in [4,7], are considered special cases of this work.

(8) Many important relationships can be derived from (14)

If \( n_j = 2m_j \), \( \frac{x}{a} = \frac{\sin \frac{\xi}{2}}{\sin \frac{\alpha}{2}}, \quad \frac{y}{a} = \frac{\sin \frac{\eta}{2}}{2} \); and if \( n_j = 2m_j + 1 \),

\[
\frac{x}{a} = \frac{\tan \frac{\xi}{2}}{\tan \frac{\alpha}{2}}, \quad \frac{y}{a} = \frac{\tan \frac{\eta}{2}}{\tan \frac{\alpha}{2}},
\]

we have the following SFIEs

\[
\sum_{j=0}^{l} u_j F_{j,\ell} \frac{\alpha}{\pi} \left[ \ln \frac{1}{2\sin \left( \frac{\xi - \eta}{2} \right)} + d \right] \psi_j(\xi) d\xi = h_k(\eta) \] (40)

The above system leads to the following SRs

\[
\sum_{j=0}^{l} u_j F_{j,\ell} \frac{\alpha}{\pi} \left[ \ln \frac{1}{2\sin \left( \frac{\xi - \eta}{2} \right)} + d \right] T_{2n_j} \left( \frac{\sin \frac{\eta}{2}}{\sin \frac{\alpha}{2}} \right) \cos \left( \frac{\eta}{2} \right) d\eta
\]

\[
= \begin{cases} 
\pi P_{\ell} \left( \ln \frac{2}{\sin \alpha} + d \right) & m_\ell = 0 \\
\frac{\pi}{2m_\ell} T_{m_\ell} \left( \frac{\sin \frac{\xi}{2}}{\sin \frac{\alpha}{2}} \right) & m_\ell \geq 1, \ell = 1, 2, \ldots, N
\end{cases} \] (41)

and

\[
\sum_{j=0}^{l} u_j F_{j,\ell} \frac{\alpha}{\pi} \left[ \ln \frac{1}{2\sin \left( \frac{\xi - \eta}{2} \right)} + d \right] T_{2m_j+1} \left( \frac{\tan \frac{\eta}{2}}{\tan \frac{\alpha}{2}} \right) \cos \left( \frac{\eta}{2} \right) d\eta
\]

\[
= \frac{\pi}{2m_\ell + 1} T_{2m_j+1} \left( \frac{\tan \frac{\xi}{2}}{\tan \frac{\alpha}{2}} \right) \] (42)

\[
m_\ell \geq 0
\]
(ii) Differentiating (14) with respect to \( x \), we have
\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{T_{n_j} \left( \frac{y}{a} \right)}{y-x} \frac{dy}{\sqrt{a^2 - y^2}} = \pi U_{n_{\ell} - 1} \left( \frac{x}{a} \right) \quad n_\ell \geq 1
\]
\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{dy}{(y-x)\sqrt{a^2 - y^2}} = 0 \quad (43)
\]
where \( U_{n_{\ell}} \left( \frac{x}{a} \right) \) are the CPs of the second kind.

Also (43) yields
\[
\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \cot \frac{\eta - \xi}{2} \frac{T_{n_j} \left( \frac{\tan \frac{\eta}{2}}{\tan \frac{\alpha}{2}} \right)}{\sqrt{2(\cos \eta - \cos \alpha)}} \cos \left( \frac{\eta}{2} \right) d\eta
\]
\[
= \begin{cases} 
0 & n_\ell = 0 \\
2 \cos ec \left( \frac{\alpha}{2} \right) J_{2m_{\ell} - 1} \left( \tan \frac{\xi}{2} \tan \frac{\alpha}{2} \right) & n_\ell = 2m_{\ell} \\
2 \left[ \cos ec \left( \frac{\alpha}{2} \right) J_{2m_{\ell} - 1} \left( \tan \frac{\xi}{2} \tan \frac{\alpha}{2} \right) + (-1)^n \sin \alpha \frac{1}{1 + \cos \alpha} \left[ \tan \frac{\alpha}{2} \right]^{2m_{\ell} - 2} \right] & n_\ell = 2m_{\ell} - 1
\end{cases}
\]
\[
(44)
\]
\[
\frac{1}{2} \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{\sec \frac{\eta}{2} \cot \frac{\eta - \xi}{2}}{\sqrt{2(\cos \eta - \cos \alpha)}} T_{n_j} \left( \frac{\tan \frac{\eta}{2}}{\tan \frac{\alpha}{2}} \right) d\eta
\]
\[
= \begin{cases} 
\cos ec \left( \frac{\alpha}{2} \right) \sec^2 \left( \frac{\xi}{2} \right) J_{n_{\ell} - 1} \left( \tan \frac{\xi}{2} \tan \frac{\alpha}{2} \right) & n \geq 1 \\
\sec \left( \frac{\alpha}{2} \right) \tan \left( \frac{\xi}{2} \right) & n_\ell = 0
\end{cases}
\]
\[
(45)
\]

(9) The mixed integral equation with Carleman kernel can be established from this work by using the following relation
\[
\ln |x - y| = h(x, y)|x - y|^{-\nu} \quad 0 < \nu < 1 \quad (46)
\]
where \[ h(x, y) = |x - y|^{\nu} \ln|x - y| \] is a smooth function.

The importance of Carleman kernel came from the work of Arutiunion [12] who has shown that, the contact problem of nonlinear theory of plasticity, in its first approximation reduce to FIE of the first kind with Carleman kernel.

References

Krein's method and Volterra – Fredholm integral equation


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