Inequalities for Analytic Functions Defined by Certain Fractional Derivative Operator

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Abstract

The object of the present paper is to give an application of the fractional derivative operator for p-valent functions in the open unit disk to the differential inequalities.

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1 Introduction and Preliminaries

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}$$

which are analytic and p-valent in the open unit disk $U = \{ z : |z| < 1 \}$.

Let $2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in U$ by, (see Srivastava and Karlsson [6])

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n$$

(1.2)
where \((\lambda)_n\) is the Pochhammer symbol defined, in terms of the Gamma function, by
\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1 & \text{when } n = 0, \\
\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & \text{when } n \in \mathbb{N}.
\end{cases}
\] (1.3)
for \(\lambda \neq 0, -1, -2, \ldots\).

We recall the following definitions of fractional derivative operators which were used by Owa [4], (see also [5]) as follows:

**Definition 1.1** The fractional derivative operator of order \(\lambda\) is defined by
\[
D_\lambda^z f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi
\] (1.4)
where \(0 \leq \lambda < 1\), \(f(z)\) is analytic function in a simply- connected region of the \(z\)-plane containing the origin, and the multiplicity of \((z-\xi)^{-\lambda}\) is removed by requiring \(\log(z-\xi)\) to be real when \(z-\xi > 0\).

**Definition 1.2** Let \(0 \leq \lambda < 1\), and \(\mu, \eta \in \mathbb{R}\). Then, in terms of the familiar Gauss hypergeometric function \(_2F_1\), the generalized fractional derivative operator \(J^\lambda_{0,z}\) is
\[
J^\lambda_{0,z} f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi) _2F_1(\mu-\lambda,1-\eta;1-\lambda;1-\xi/z) \right)
\] (1.5)
where \(f(z)\) is analytic function in a simply- connected region of the \(z\)-plane containing the origin, with the order \(f(z) = O(|z|^\varepsilon)\), \(z \to 0\), where \(\varepsilon > \max\{0,\mu-\eta\} - 1\) and the multiplicity of \((z-\xi)^{-\lambda}\) is removed by requiring \(\log(z-\xi)\) to be real when \(z-\xi > 0\).

**Definition 1.3** Under the hypotheses of Definition 1.2, the fractional derivative operator \(J^\lambda_{0,z}^{\mu+m,\eta+m}\) of a function \(f(z)\) is defined by
\[
J^\lambda_{0,z}^{\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J^\lambda_{0,z} f(z)
\] (1.6)

Notice that
\[
J^\lambda_{0,z}^{\lambda,\eta} f(z) = D_\lambda f(z), \quad 0 \leq \lambda < 1
\] (1.7)

Motivated by the investigation of the fractional derivative operator on the class of analytic functions [1, 2, 5, 7], we define a modification of the fractional derivative operator \(M^\lambda_{0,z}^{\mu,\eta}\) by
\[
M^\lambda_{0,z}^{\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J^\lambda_{0,z}^{\mu,\eta} f(z)
\] (1.8)
Inequalities for analytic functions

for \( f(z) \in \mathcal{A}(p) \) and \( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N} \). Then it is observed that \( M_{0,z}^{\lambda,\mu,\eta} f(z) \) maps \( \mathcal{A}(p) \) onto itself as follows:

\[
M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n}
\] (1.9)

where

\[
\delta_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}
\] (1.10)

It is easily verified from (1.9) that

\[
z \left( M_{0,z}^{\lambda,\mu,\eta} f(z) \right)' = (p-\mu)M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) + \mu M_{0,z}^{\lambda,\mu,\eta} f(z)
\] (1.11)

This identity plays a critical role in obtaining information about functions defined by use of the fractional derivative operator. Our results in this paper will rely heavily on the identity.

Notice that

\[M_{0,z}^{0,0,\eta} f(z) = f(z)\]

and

\[M_{0,z}^{1,1,\eta} f(z) = \frac{zf'(z)}{p}\]

In our present investigation, we shall use the method of differential inequalities introduced by Miller and Mocanu [3] to derive certain properties of fractional derivative operator.

To prove our results, we need the following lemmas.

Lemma 1.4 [3] Let \( w(z) = b_p z^p + b_{p+1} z^{p+1} + \ldots \) \( (p \in \mathbb{N}) \) be regular in the unit disk \( \mathcal{U} \) with \( w(0) \neq 0 \) \( (z \in \mathcal{U}) \). If \( z_0 = r_0 e^{i\theta} \) \( (0 < r_0 < 1) \) and \( |w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| \), then

\[z_0 w'(z_0) = \xi w(z_0)\] (1.12)

and

\[
\text{Re} \left\{ 1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right\} \geq \xi
\] (1.13)

where \( \xi \) is real and \( \xi \geq p \geq 1 \).
Lemma 1.5 [3] Let \( w(z) = a + w_k z^k + \ldots \) be regular in the unit disk \( \mathcal{U} \) with \( w(z) \neq a \) and \( k \geq 1 \). If \( z_0 = r_0 e^{i\theta} \) (\( 0 < r_0 < 1 \)) and \( |w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| \), then

\[
    z_0 w'(z_0) = \xi w(z_0)
\]

and

\[
    \text{Re} \left\{ 1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right\} \geq \xi
\]

where \( \xi \) is real number and

\[
    \xi \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}
\]

2 Inequalities Associated With Fractional Derivative Operator

We begin with the following definition for a class of functions which we require in our first result.

Definition 2.1 Let \( \Phi \) be the set of complex-valued functions \( \phi(r, s, t) \);

\[ \phi(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C} \quad (\mathbb{C} \text{ is a complex plane}) \]

such that

1. \( \phi(r, s, t) \) is continuous in a domain \( D \subset \mathbb{C}^3 \);
2. \((0, 0, 0) \in D \) and \( |\phi(0, 0, 0)| < 1 \);
3. \[ \phi \left( e^{i\theta}, \left( \frac{\xi - \mu}{p - \mu} \right) e^{i\theta}, \frac{\mu + 1 - 2\xi}{(p - \mu)(p - \mu - 1)} e^{i\theta} + M \right) \geq 1 \]

whenever,

\[ \left( e^{i\theta}, \left( \frac{\xi - \mu}{p - \mu} \right) e^{i\theta}, \frac{\mu + 1 - 2\xi}{(p - \mu)(p - \mu - 1)} e^{i\theta} + M \right) \in D \]

with \( \text{Re} \{ e^{-i\theta} M \} \geq \xi (\xi - 1) \), for all \( \theta \in \mathbb{R} \) and for all \( \xi \geq p \geq 1 \).

Theorem 2.2 Let \( \phi(r, s, t) \in \Phi \) and let \( f(z) \in A(p) \) satisfying

\[
    \left( M_{0, z}^{\lambda, \mu, \eta} f(z), M_{0, z}^{\lambda+1, \mu+1, \eta+1} f(z), M_{0, z}^{\lambda+2, \mu+2, \eta+2} f(z) \right) \in D \subset \mathbb{C}^3
\]
| φ(M_0^λ,μ,ηf(z), M_0^{λ+1,μ+1,η+1}f(z), M_0^{λ+2,μ+2,η+2}f(z)) | < 1 \quad (2.2) \\

(\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; z \in U) \\

Then we have

| M_0^{λ,μ,η}f(z) | < 1 \quad (2.3) \\

**Proof.** We define the function \( w(z) \) by

\[
M_0^{λ,μ,η}f(z) = w(z) \quad (2.4)
\]

(\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; z \in U) \\

for \( f(z) \in A(p) \). Then it follows that \( w(z) \in A(p) \) and \( w(z) \neq 0 \) (\( z \in U \)). 

With the aid of the identity (1.11), we have

\[
M_0^{λ+1,μ+1,η+1}f(z) = \frac{1}{(p-\mu)} \left\{ zw'(z) - \mu w(z) \right\} \quad (2.5)
\]

and

\[
M_0^{λ+2,μ+2,η+2}f(z) = \frac{1}{(p-\mu)(p-\mu-1)} \left\{ \mu(\mu+1)w(z) - 2\mu zw'(z) + z^2 w''(z) \right\} \quad (2.6)
\]

Suppose that \( z_0 = r_0 e^{i\theta} \) (\( 0 < r_0 < 1, \theta \in \mathbb{R} \)) and

\[
|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = 1 \quad (2.7)
\]

Then, letting \( w(z_0) = e^{i\theta} \) and using (1.12) of Lemma 1.4, we obtain

\[
M_0^{λ,μ,η}f(z_0) = w(z_0) = e^{i\theta}, \quad (2.8)
\]

\[
M_0^{λ+1,μ+1,η+1}f(z_0) = \left( \frac{\xi - \mu}{p - \mu} \right) w(z_0) = \left( \frac{\xi - \mu}{p - \mu} \right) e^{i\theta}, \quad (2.9)
\]

and

\[
M_0^{λ+2,μ+2,η+2}f(z_0) = \left( \frac{\mu(\mu + 1 - 2\xi)}{(p - \mu)(p - \mu - 1)} \right) \left\{ \mu(\mu + 1 - 2\xi) e^{i\theta} + z_0^2 w''(z_0) \right\} \\
= \frac{\mu(\mu + 1 - 2\xi) e^{i\theta} + M}{(p - \mu)(p - \mu - 1)} \quad (2.10)
\]
where \( M = z_0^2 w''(z_0) \) and \( \xi \geq p \geq 1 \).

Further, an application of (1.13) in Lemma 1.4 gives

\[
\text{Re} \left\{ \frac{z_0 w''(z_0)}{w'(z_0)} \right\} = \text{Re} \left\{ \frac{z_0^2 w''(z_0)}{\xi e^{i\theta}} \right\} \geq \xi - 1 \quad (2.11)
\]

or

\[
\text{Re} \left\{ e^{-i\theta} M \right\} \geq \xi (\xi - 1) \quad (\theta \in \mathbb{R}, \xi \geq 1) \quad (2.12)
\]

Since \( \phi(r, s, t) \in \Phi \), we also have

\[
\left| \phi(M_{0,z}^{\lambda,\mu,\eta} f(z), M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z), M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)) \right| =
\left| \phi(e^{i\theta}, \frac{\xi - \mu}{p - \mu} e^{i\theta}, \frac{\mu(\mu + 1 - 2\xi) e^{i\theta} + M}{(p - \mu)(p - \mu - 1)} \right| > 1 \quad (2.13)
\]

which contradicts the condition (2.2) of Theorem 2.2. Therefore, we conclude that

\[
|w(z)| = |M_{h,z}^{\lambda,\mu,\eta} f(z)| < 1 \quad (2.14)
\]

This completes the proof of Theorem 2.2.

In order to prove our next result, we need the following:

**Definition 2.3** Let \( H \) be the set of complex-valued functions \( h(r, s, t) \);

\[
h(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C} \quad (\mathbb{C} \text{ is a complex plane})
\]

such that

1. \( h(r, s, t) \) is continuous in a domain \( D \subset \mathbb{C}^3 \);
2. \( (1, 1, 1) \in D \) and \( |h(1, 1, 1)| < J \), \( (J > 1) \);
3. \[
\left| h(J e^{i\theta}, \frac{\xi - 1 + (p - \mu) J e^{i\theta}}{p - \mu - 1}, \frac{1}{p - \mu - 2} \left\{ \xi - 2 + (p - \mu) J e^{i\theta} + \frac{\xi - \xi^2 + (p - \mu) J e^{i\theta} + L}{\xi - 1 + (p - \mu) J e^{i\theta}} \right\} ) \right| >
\]

whenever,

\[
\left( J e^{i\theta}, \frac{\xi - 1 + (p - \mu) J e^{i\theta}}{p - \mu - 1}, \frac{1}{(p - \mu - 2)} \left\{ \xi - 2 + (p - \mu) J e^{i\theta} + \frac{\xi - \xi^2 + (p - \mu) J e^{i\theta} + L}{\xi - 1 + (p - \mu) J e^{i\theta}} \right\} \right) \in D
\]

with \( \text{Re}\{L\} \geq \xi(\xi - 1) \), for all \( \theta \in \mathbb{R} \) and for all \( \xi \geq \frac{J-1}{J+1} \).
Theorem 2.4 Let \( h(r, s, t) \in H \) and let \( f(z) \in A(p) \) satisfying

\[
\left( \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)} \right) \in D \subset C^3 \tag{2.15}
\]

and

\[
\left| h \left( \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}, \frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}, \frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)} \right) \right| < J \tag{2.16}
\]

\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; J > 1; z \in U)\)

Then we have

\[
\left| \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \right| < J \tag{2.17}
\]

**Proof.** We define the function \( w(z) \) by

\[
\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} = w(z) \tag{2.18}
\]

\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; z \in U)\)

for \( f(z) \in A(p) \). Then it follows that \( w(z) \) is either analytic or meromorphic in \( U \), \( w(0) = 1 \) and \( w(z) \neq 1 \). With the aid of the identity (1.11), we have

\[
\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} = \frac{1}{(p - \mu - 1)} \left\{ \frac{zw'(z)}{w(z)} + (p - \mu)w(z) - 1 \right\} \tag{2.19}
\]

and

\[
\frac{M_{0,z}^{\lambda+3,\mu+3,\eta+3} f(z)}{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)} = \frac{1}{(p - \mu - 2)} \left\{ \frac{zw'(z)}{w(z)} + (p - \mu)w(z) - 2 \right. \\
+ \left. \frac{(p - \mu)zw''(z)}{w(z)} + \frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left( \frac{zw'(z)}{w(z)} \right)^2 \right\} \tag{2.20}
\]

Suppose that \( z_0 = r_0 e^{i\theta} \) \((0 < r_0 < 1, \theta \in \mathbb{R}) \) and

\[
|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = J \tag{2.21}
\]
Then, letting \( w(z_0) = J e^{i\theta} \) and using Lemma 1.5, with \( a = k = 1 \), we see that

\[
\frac{M_{\lambda+2,\mu+2,\eta+2}^{\lambda+1,\mu+1,\eta+1} f(z_0)}{M_{\lambda+1,\mu+1,\eta+1} f(z_0)} = \frac{1}{(p - \mu - 1)} \left\{ \xi - 1 + (p - \mu) J e^{i\theta} \right\}, \tag{2.22}
\]

and

\[
\frac{M_{\lambda+3,\mu+3,\eta+3}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{\lambda+2,\mu+2,\eta+2} f(z)} = \frac{1}{(p - \mu - 1)} \left\{ \xi + (p - \mu) J e^{i\theta} - 2 + \frac{(p - \mu)\xi J e^{i\theta} + \xi - \xi^2 + L}{\xi + (p - \mu) J e^{i\theta} - 1} \right\} \tag{2.23}
\]

where \( L = \frac{w''(z_0)}{w(z_0)} \) and \( \xi \geq \frac{J - 1}{J + 1} \).

Further, an application of (1.15) in Lemma 1.5 gives

\[
\text{Re}\{L\} \geq \xi (\xi - 1) \tag{2.24}
\]

Since \( h(r, s, t) \in H \), we have

\[
\left| h \left( M_{\lambda+1,\mu+1,\eta+1}^{\lambda+2,\mu+2,\eta+2} f(z) f(z), \frac{M_{\lambda+2,\mu+2,\eta+2}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{\lambda+1,\mu+1,\eta+1} f(z)} \right) \right| =
\left| h \left( J e^{i\theta} \xi - 1 + (p - \mu) J e^{i\theta} \right), \frac{1}{p - \mu - 2} \left\{ \frac{\xi - 2 + (p - \mu) J e^{i\theta} + L}{\xi - 1 + (p - \mu) J e^{i\theta}} \right\} \right| > J \tag{2.25}
\]

which contradicts the condition (2.16) of Theorem 2.4. Therefore, we conclude that

\[
|w(z)| = \left| \frac{M_{\lambda+1,\mu+1,\eta+1}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{\lambda+2,\mu+2,\eta+2} f(z)} \right| < J \tag{2.26}
\]

This completes the proof of Theorem 2.4.

**References**


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