Fuzzy b-Generalized Homeomorphism in Fuzzy Topological Spaces

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Abstract

In this paper, some results on fbg-closed sets and fbg-continuous mappings are obtained. Fbg-neighbourhood, fbgq-neighbourhood are introduced and their basic properties are studied. New spaces namely $fbgT^*_1/2$ are introduced and characterized. The concept of fbg-closed, fbg*-closed, fuzzy bg-homeomorphism, fuzzy bg*-homeomorphism, contra fb-continuous and contra fb-closed mappings are introduced and studied.

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1 Introduction

With the introduction of fuzzy sets by Zadeh [7] and fuzzy topology by Chang [6], the theory of fuzzy topological spaces was subsequently developed by several authors by considering the basic concepts of general topology. Fb-open sets, fbg-closed sets were introduced by Benchalli et al [2,4].

The aim of this paper is to introduce and study the notion of fbg-neighbourhood, fbgq-neighbourhood and new spaces namely $f_{bg}T^{*}_{1/2}$ fuzzy topological spaces. Further, the concept of fbg-closed, fbg*-closed, fuzzy bg-homeomorphism, fuzzy bg*-homeomorphism, contra fb-continuous and contra fb-closed mappings are introduced and studied.

2 Preliminary Notes

Throughout this paper, $(X,\tau)$, $(Y,\sigma)$ and $(Z,\rho)$ (or simply $X$, $Y$ and $Z$) always mean fuzzy topological spaces. For a fuzzy set $A$ of $(X,\tau)$, $Cl(A)$ and $Int(A)$ denote the closure and interior of $A$ respectively. A fuzzy subset $A$ of $X$ is said to be fb-open (fb-closed) if $A \leq ClInt(A) \lor IntCl(A)$ ($A \leq (IntCl(A) \land ClInt(A))$). The family of fb-open sets is denoted by $bO(X)$.

The intersection of all fb-closed sets containing $A$ is called fb-closure of $A$ and is denoted by $bCl(A)$ and the union of all fb-open sets contained in $A$ is called fb-interior of $A$ and is denoted by $bInt(A)$.

Definition 2.1 A mapping $f: (X,\tau) \rightarrow (Y,\sigma)$ is said to be

(a) f-continuous [6] if $f^{-1}(A)$ is f-open in $X$, for each f-open set $A$ in $Y$.
(b) fb-continuous [3] if $f^{-1}(A)$ is fb-open in $X$, for each fb-open set $A$ in $Y$.
(c) fb*-continuous [3] if $f^{-1}(A)$ is fb-open in $X$, for each fb-open set $A$ in $Y$.
(d) fb-open [3] if for every fb-open $A$ in $X$, $f(A)$ is fb-open set in $Y$.
(e) fab-closed [5] if $f(B) \leq bInt(A)$, whenever $A$ is fb-open of $Y$, $B$ is fb-closed set in $X$ and $f(B) \leq A$. (f) fbg-continuous [4] if $f^{-1}(A)$ is fbg-open (fbg-closed) in $X$, for each f-open (f-closed) set $A$ in $Y$.

Definition 2.2 [4] A fuzzy set $A$ in a fts $(X,\tau)$ is called

(a) fbg-closed iff $bCl(A) \leq B$, whenever $A \leq B$ and $B$ is fb-open in $X$.
(b) fbg-open iff $B \leq bInt(A)$, whenever $B \leq A$ and $B$ is fb-closed in $X$.

Definition 2.3 [1] A fts $(X,\tau)$ is called is called a fuzzy $gT^{*}_{1/2}$ space (briefly $fgT^{*}_{1/2}$ space) if every fg-closed set in $X$ is f-closed.

3 Fuzzy b-generalized closed sets

Some of the results on fbg-closed sets are in proved in [3]. In this section few more results on fbg-closed sets are proved.
Theorem 3.1 A fuzzy set $A$ of a fts $(X, \tau)$ is called fb-open iff $B \leq b\text{Int}(A)$, whenever $B$ is fb-closed and $B \leq A$.

Proof. Suppose $A$ is fb-open in $X$. Then $1 - A$ is fb-closed in $X$. Let $B$ be a fb-closed set in $X$ such that $B \leq A$. Then $1 - A \leq 1 - B$, $1 - B$ is fb-open set in $X$. Since $1 - A$ is fb-closed, $b\text{Cl}(1 - A) \leq 1 - B$, which implies $1 - b\text{Int}(A) \leq 1 - B$. Thus $B \leq b\text{Int}(A)$.

Conversely, assume that $B \leq b\text{Int}(A)$, whenever $B \leq A$ and $B$ is fb-closed set in $X$. Then $1 - b\text{Int}(A) \leq 1 - B = C$, where $C$ is fb-open set in $X$. Hence $b\text{Cl}(1 - A) \leq C$, which implies $1 - A$ is fb-closed. Therefore $A$ is fb-open.

Theorem 3.2 A finite union of fb-open sets is a fb-open set.

However, intersection of any two fb-open sets need not be fb-open, as shown in the following example.

Example 3.3 Let $X = \{a, b, c\}$ and $\tau = \{0, 1, A\}$, where $A = \{(a, 1), (b, 0.5), (c, 0.3)\}$. Let $C = \{(a, 1), (b, 0.6), (c, 0)\}$ and $B = \{(a, 0), (b, 0.6), (c, 0)\}$ be fuzzy sets in $X$. Then $C$ and $B$ are fb-open in $X$ and hence $C$ and $B$ are fb-open in $X$. But $C \land B = \{(a, 0), (b, 0.4), (c, 0)\}$ is not fb-open in $X$, since $b\text{Cl}(C \lor B) = 1 \leq C \lor B$.

Theorem 3.4 Finite intersection of fb-closed sets is a fb-closed set.

However, union of two fb-closed sets need not be a fb-closed set as shown in the following example.

Example 3.5 Let $X = \{a, b, c\}$ and $\tau = \{0, 1, A\}$, where $A = \{(a, 1), (b, 0.5), (c, 0)\}$. Let $C = \{(a, 0), (b, 0.4), (c, 1)\}$ and $B = \{(a, 1), (b, 0.4), (c, 1)\}$ be fuzzy sets in $X$. Then $C$ and $B$ are fb-closed set in $X$ and hence $C$ and $B$ are fb-closed in $X$. But $C \lor B = \{(a, 1), (b, 0.4), (c, 1)\}$ is not fb-closed set in $X$, since $b\text{Cl}(C \lor B) = 1 \leq C \lor B$.

Theorem 3.6 Every f-continuous map is fb-continuous.

However, converse need not be true as shown in the following example.

Example 3.7 Let $X = \{a, b, c\}$ and $Y = \{l, m\}$. Let the fuzzy sets $A$ and $B$ be defined as follows: $A = \{(a, 1), (b, 0), (c, 0)\}$, $B = \{(l, 0), (m, 1)\}$. Consider $\tau = \{0, 1, A\}$ and $\rho = \{0, 1, B\}$. Define $f : X \rightarrow Y$ as $f(a) = f(c) = m$ and $f(b) = l$. Then $f$ is fb-continuous, but not f-continuous, since for the open fuzzy set $B \in \rho$, $f^{-1}(B) \notin \tau$.

Theorem 3.8 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be fb-continuous and $g : (Y, \sigma) \rightarrow (Z, \rho)$ be fg-continuous. Then $gof$ is fb-continuous if $Y$ is $fT_{1/2}$ space.
Definition 3.9 Let $A$ be a fuzzy set in fts $X$ and $x_p$ be a fuzzy point of $X$, then $A$ is called fuzzy b-generalized neighbourhood (briefly fbg-neighbourhood) of $x_p$ if and only if there exists a fbg-open set $B$ of $X$ such that $x_p \in B \leq A$.

Definition 3.10 Let $A$ be a fuzzy set in fts $X$ and $x_p$ be a fuzzy point of $X$, then $A$ is called fuzzy b-generalized q-neighbourhood (briefly fbgq-neighbourhood) of $x_p$ if and only if there exist a fbg-open set $B$ such that $x_p q B \leq A$.

The proof of the following three theorems are straight forward.

Theorem 3.11 $A$ is fbg-open set in $X$ if and only if for each fuzzy point $x_p \in A$, $A$ is a fbg-neighbourhood of $x_p$.

Theorem 3.12 If $A$ and $B$ are fbg-neighbourhood of $x_p$ then $A \land B$ is also a fbg-neighbourhood of $x_p$.

Theorem 3.13 Let $A$ be a fuzzy set of a fts $X$. Then a fuzzy point $x_p \in bCl(A)$ if and only if every fbgq-neighbourhood of $x_p$ is quasi-coincident with $A$.

Definition 3.14 A fts $(X, \tau)$ is called a fuzzy $bT_{1/2}^*$ space (in short $fbT_{1/2}^*$ space) if every fbg-closed set in $X$ is fuzzy closed.

Theorem 3.15 A fts $(X, \tau)$ is $fbT_{1/2}^*$ space if and only if every fuzzy set in $(X, \tau)$ is both fuzzy open and fbg-open.

Remark 3.16 A fts $(X, \tau)$ is called a fuzzy $bT_{1/2}^*$ space (in short $fbT_{1/2}^*$ space) if every fbg-open set in $X$ is fuzzy open.

Definition 3.17 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy bg-open (briefly fbg-open) if the image of every f-open set in $X$, is fbg-open in $Y$.

Definition 3.18 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy bg-closed (briefly fbg-closed) if the image of every f-closed set in $X$ is fbg-closed in $Y$.

Definition 3.19 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy bg$^*$-open (briefly fbg$^*$-open) if the image of every fbg-open set in $X$ is fbg-open in $Y$.

Definition 3.20 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy bg$^*$-closed (briefly fbg$^*$-closed) if the image of every fbg-closed set in $X$ is fbg-closed in $Y$.

Remark 3.21 (i) Every fbg$^*$-closed mapping is fbg-closed.
(ii) Every fbg$^*$-closed mapping is fgb$^*$-closed.
The proof of the following theorems on composition of mappings are straight forward.

**Theorem 3.22** If \( f : (X, \tau) \to (Y, \sigma) \) is f-closed and \( g : (Y, \sigma) \to (Z, \rho) \) is fb-g-closed, then \( \text{gof} \) is fb-g-closed.

**Theorem 3.23** If \( f : (X, \tau) \to (Y, \sigma) \) is a fb-open map and \( Y \) is fbT\(_{1/2}\) space, then \( f \) is a f-closed map.

**Theorem 3.24** If \( f : (X, \tau) \to (Y, \sigma) \) be a fb-open map and \( X \) is fbgT\(_{1/2}\) space, then \( f \) is a fbg-closed map.

**Theorem 3.25** A mapping \( f : (X, \tau) \to (Y, \sigma) \) is fb-closed iff for each fuzzy set \( A \) in \( Y \) and f-open set \( B \) such that \( f^{-1}(A) \subseteq B \), there is a fb-open set \( C \) of \( Y \) such that \( A \subseteq C \) and \( f^{-1}(C) \subseteq B \).

**Proof.** Suppose \( f \) is fb-closed map. Let \( A \) be a fuzzy set of \( Y \), and \( B \) be an f-open set of \( X \), such that \( f^{-1}(A) \subseteq B \). Then \( C = 1 - f(1-B) \) is a fb-open set in \( Y \) such that \( A \subseteq C \) and \( f^{-1}(C) \subseteq B \).

Conversely, suppose that \( F \) is a f-closed set of \( X \). Then \( f^{-1}(1 - f(F)) \subseteq 1 - F \), and \( 1 - F \) is f-open set. By hypothesis, there is a fb-open set \( C \) of \( Y \) such that \( 1 - f(1-F) \subseteq C \) and \( f^{-1}(C) \subseteq 1 - F \). Therefore \( F \subseteq 1 - f^{-1}(C) \). Hence \( 1-C \subseteq f(C) \subseteq f(1-f^{-1}(C)) \subseteq 1-C \), which implies \( f(F) = 1-C \). Since \( 1-C \) is fb-closed set, \( f(F) \) is fb-closed set and thus \( f \) is a fb-closed map.

**Theorem 3.26** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \rho) \) are fb-closed maps and \( Y \) is fbT\(_{1/2}\) space, then \( \text{gof} : X \to Z \) is fb-closed.

**Theorem 3.27** If \( A \) is fb-closed in \( X \) and \( f : X \to Y \) is bijective, fb-irresolute and fb-closed, then \( f(A) \) is fb-closed in \( Y \).

**Proof.** Let \( f(A) \subseteq B \) where \( B \) is fb-open in \( Y \). Since \( f \) is fb-irresolute, \( f^{-1}(B) \) is fb-open containing \( A \). Hence \( bCl(A) \subseteq f^{-1}(B) \) as \( A \) is fb-closed. Since \( f \) is fb-closed, \( f(bCl(A)) \) is fb-closed set contained in the fb-open set \( B \), which implies \( bCl(f(bCl(A))) \subseteq B \) and hence \( bCl(f(A)) \subseteq B \). So \( f(A) \) is fb-closed in \( Y \).

**Theorem 3.28** If \( f : (X, \tau) \to (Y, \sigma) \) is fb-closed and \( g : (Y, \sigma) \to (Z, \rho) \) is fb*-closed, then \( \text{gof} \) is fb*-closed.

**Theorem 3.29** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \rho) \) are fb*-closed maps, then \( \text{gof} : X \to Z \) is fb*-closed.
Theorem 3.30 Let \( f : (X, \tau) \to (Y, \sigma), g : (Y, \sigma) \to (Z, \rho) \) be two maps such that \( g \circ f : X \to Z \) is fbg-closed.

(a) If \( f \) is f-continuous and surjective, then \( g \) is fbg-closed.
(b) If \( g \) is fbg-irresolute and injective, then \( f \) is fbg-closed.

Proof. (a) Let \( F \) be f-closed in \( Y \). Then \( f^{-1}(F) \) is f-closed in \( X \), as \( f \) is f-continuous. Since \( g \circ f \) is fbg-closed map and \( f \) is surjective, \((g \circ f)(f^{-1}(F)) = g(F) \) is fbg-closed in \( Z \). Hence \( g : Y \to Z \) is fbg-closed.

(b) Let \( F \) be a f-closed in \( X \). Then \((g \circ f)(F) \) is fbg-closed in \( Z \). Since \( g \) is fbg-irresolute and injective \( g^{-1}(g \circ f)(F) = f(F) \) is fbg-closed in \( Y \). Hence \( f \) is a fbg-closed.

Theorem 3.31 Let \( f : (X, \tau) \to (Y, \sigma), g : (Y, \sigma) \to (Z, \rho) \) be two maps such that \( g \circ f : X \to Z \) is fbg*-closed map.

(a) If \( f \) is fbg-continuous and surjective, then \( g \) is fbg-closed.
(b) If \( g \) is fbg-irresolute and injective, then \( f \) is fbg*-closed.

Theorem 3.32 Let \( f : (X, \tau) \to (Y, \sigma) \). Then the following statements are equivalent.

(a) \( f \) is fbg-irresolute.
(b) for every fbg-closed set \( A \) in \( Y \), \( f^{-1}(A) \) is fbg-closed in \( X \).
(c) for every fuzzy point \( x_p \) of \( X \) and every fbg-open \( A \) of \( Y \) such that \( f(x_p) \in A \), there exist a fbg-open set such that \( x_p \in B \) and \( f(B) \leq A \).
(d) for every fuzzy point \( x_p \) of \( X \) and every fbg-neighbourhood \( A \) of \( f(x) \), \( f^{-1}(A) \) is a fbg-neighbourhood of \( x_p \).
(e) for every fuzzy point \( x_p \) of \( X \) and every fbg-neighbourhood \( A \) of \( f(x_p) \), there is a fbg-neighbourhood \( B \) of \( x_p \) such that \( f(B) \leq A \).
(f) for every fuzzy point \( x_p \) of \( X \) and every fbg-open set \( A \) of \( Y \) such that \( f(x_p) \cap A \), there exists a fbg-open set \( B \) of \( X \) such that \( x_p \in B \) and \( f(B) \leq A \).
(g) for every fuzzy point \( x_p \) of \( X \) and every fbgq-neighbourhood \( A \) of \( f(x_p) \), \( f^{-1}(A) \) is a fbgq-neighbourhood of \( x_p \).
(h) for every fuzzy point \( x_p \) of \( X \) and every fbgq-neighbourhood \( B \) of \( x_p \), there exists a fbgq-neighbourhood \( A \) of \( x_p \), such that \( f(B) \leq A \).

Proof. (a)\( \Rightarrow \) (b) Obvious.

(b)\( \Rightarrow \) (a) Let \( A \) be a fbg-closed set in \( Y \) which implies \( 1 - A \) is fbg-open in \( Y \). \( f^{-1}(1 - A) \) is fbg-open in \( X \) implies \( f^{-1}(A) \) is fbg-closed in \( X \). Hence \( f \) is fbg-irresolute.

(c)\( \Rightarrow \) (a) Let \( A \) be a fbg-open set in \( Y \) and \( x_p \in f^{-1}(A) \) which implies \( f(x_p) \in A \). Then there exist a fbg-open set \( B \) in \( X \) such that \( x_p \in B \) and \( f(B) \leq A \). Hence \( x_p \in B \leq f^{-1}(A) \). Hence \( f^{-1}(A) \) is fbg-open in \( X \). Hence \( f \) is fbg-irresolute.

(a)\( \Rightarrow \) (d) Obvious.
(d)⇒(a) Obvious.
(d)⇒(e) Let \( x_p \) be a fuzzy point of \( X \) and \( A \) be a fbg-neighbourhood of \( f(x_p) \). Then \( B = f^{-1}(A) \) is a fbg-neighbourhood of \( x_p \) and \( f(B) = f(f^{-1}(A)) \leq A \).
(e)⇒(c) Let \( x_p \) be a fuzzy point of \( X \) and \( A \) be a fbg-open set such that \( f(x_p) \leq A \). Then \( A \) is a fbg-neighbourhood of \( f(x_p) \). Hence there is fbg-neighbourhood \( B \) of \( x_p \) in \( X \) such that \( x_p \in B \) and \( f(B) \in A \). Hence there is fbg-open set \( C \) in \( X \) such that \( x_p \in C \leq B \) and \( f(C) \leq f(B) \leq A \).
(a)⇒(f) Let \( x_p \) be a fuzzy point of \( X \) and \( A \) be a fbg-open set in \( Y \) such that \( f(x_p)qA \). Let \( B = f^{-1}(A) \). \( B \) is a fbg-open set in \( X \), such that \( x_pqB \) and \( f(B) = f(f^{-1}(A)) \leq A \).
(f)⇒(a) Let \( A \) be a fuzzy open set in \( Y \) and \( x_p \in f^{-1}(A) \). Clearly \( f(x_p) \in A \).
\((x_p)^C = 1 - x_p(x)\). Then \( f(1 - x_p)qA \). Hence there exists a fbg-open set \( B \) of \( X \) such that \((1 - x_p)qB \) and \( f(B) \leq A \). Now \((1 - x_p)qB \Rightarrow (1 - x_p)B(x) = 1 - p + B(x) > 1 \Rightarrow B(x) > p \Rightarrow x_p \in B \). Thus \( x_p \in B \leq f^{-1}(A) \). Hence \( f^{-1}(A) \) is fbg-open in \( X \).
(f)⇒(g) Let \( x_p \) be a fuzzy point of \( X \) and \( A \) be fbgq-neighbourhood of \( f(x_p) \). Then there is fbg-open set \( C \) in \( Y \) such that \( x_pqC \leq A \). By hypothesis there is a fbg-open set \( B \) of \( X \) such that \( x_pqB \) and \( f(B) \leq C \). Thus \( x_pqB \leq f^{-1}(C) \leq f^{-1}(A) \). Hence \( f^{-1}(A) \) is a fbgq-neighbourhood of \( x_p \).
(h)⇒(f) Let \( x_p \) be a fuzzy point of \( X \) and \( A \) be fbg-open in \( Y \) such that \( f(x_p)qA \). Then \( A \) is fbgq-neighbourhood of \( f(x_p) \). So there is a fbgq-neighbourhood \( C \) of \( x_p \) such that \( f(C) \leq A \). Since \( C \) is a fbgq-neighbourhood of \( x_p \) there exists a fbg-open set \( B \) of \( X \) such that \( x_pqB \leq C \). Hence \( x_pqB \) and \( f(B) \leq A \).

4 Fbg-homeomorphism and fbg*-homeomorphism

Definition 4.1 A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called fuzzy bg-homeomorphism (briefly fb-homeomorphism) if \( f \) and \( f^{-1} \) are fbg-continuous.

Definition 4.2 A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called fuzzy bg*-homeomorphism (briefly fbg*-homeomorphism) if \( f \) and \( f^{-1} \) are fbg- irresolute.

Theorem 4.3 Every f-homeomorphism is fbg-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 4.4 Let \( X = Y = \{a, b, c\} \) and the fuzzy sets \( A, B \) and \( C \) be defined as follows. \( A = \{(a, 1), (b, 0.8)\} \), \( B = \{(a, 0.3), (b, 0.6)\} \), \( C = \{(a, 0.4), (b, 0.6)\} \). Consider \( \tau = \{0, 1, A\} \) and \( \rho = \{0, 1, B\} \). Then \( (X, \tau) \) and \( (Y, \rho) \) are fts. Define \( f : X \rightarrow Y \) by \( f(a) = a \) and \( f(b) = b \). Then \( f \) is fb-homeomorphism but not f-homeomorphism as \( A \) is open in \( X \). \( f(A) \) is not open in \( Y \). Hence \( f^{-1} : Y \rightarrow X \) is not f-continuous.
Theorem 4.5 Every fbg*-homeomorphism is fbg- homeomorphism.

Theorem 4.6 Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijective mapping. Then the following are equivalent:

(a) \( f \) is fbg-homeomorphism
(b) \( f \) is fbg-continuous and fbg-open map
(c) \( f \) is fbg-continuous and fbg-closed map

Proof. (a)\( \Rightarrow \) (b) Let \( f \) be fbg-homeomorphism. Then \( f \) and \( f^{-1} \) are fbg-continuous. To prove that \( f \) is fbg-open map, let \( A \) be a fuzzy open set in \( X \). Since \( f^{-1} : Y \to X \) is fbg-continuous, \( (f^{-1})^{-1}(A) = f(A) \) is fbg-open in \( Y \). Therefore \( f(A) \) is fbg-open in \( Y \). Hence \( f \) is fbg-open.

(b)\( \Rightarrow \) (a) Let \( f \) be fbg-open and fbg-continuous map. To prove that \( f^{-1} : Y \to X \) is fbg-continuous. Let \( A \) be a fuzzy open set in \( X \). Then \( f(A) \) is fbg-open set in \( Y \) since \( f \) is fbg-open map. Now \( (f^{-1})^{-1}(A) = f(A) \) is fbg-open set in \( Y \). Therefore \( f^{-1} : Y \to X \) is fbg-continuous. Hence \( f \) is fbg-homeomorphism.

(b)\( \Rightarrow \) (c) Let \( f \) be fbg-continuous and fbg-open map. To prove that \( f \) is fbg-closed map. Let \( B \) be a fuzzy closed set in \( X \). Then \( 1-B \) is fuzzy open set in \( X \). Since \( f \) is fbg-open map, \( f(1-B) \) is fbg-open set in \( Y \). Now \( f(1-B) = 1 - f(B) \). Therefore \( f(B) \) is fbg-closed in \( Y \). Hence \( f \) is a fbg-closed.

(c)\( \Rightarrow \) (b) Let \( f \) be fbg-continuous and fbg-closed map. To prove that \( f \) is fbg-open map. Let \( A \) be a fuzzy open set in \( X \). Then \( 1-A \) is a fuzzy closed set in \( X \). Since \( f \) is fbg-closed map, \( f(1-A) \) is fbg-closed in \( Y \). Now \( f(1-A) = 1 - f(A) \). Therefore \( f(A) \) is fbg-open in \( Y \). Hence \( f \) is fbg-open.

Theorem 4.7 Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijective function. Then the following are equivalent:

(a) \( f \) is fbg*-homeomorphism
(b) \( f \) is fbg-irresolute and fbg*-open
(c) \( f \) is fbg-irresolute and fbg*-closed

Theorem 4.8 If \( f : (X, \tau) \to (Y, \sigma) \) is fbg-homeomorphism and \( g : (Y, \sigma) \to (Z, \rho) \) is fbg-homeomorphism and \( Y \) is \( f b T_{1/2} \) space, then \( g f : X \to Z \) is fbg-homeomorphism.

Theorem 4.9 If \( f : (X, \tau) \to (Y, \sigma) \), \( g : (Y, \sigma) \to (Z, \rho) \) are fbg* homeomorphism then \( g f : X \to Z \) is fbg* homeomorphism.

Definition 4.10 A map \( f : (X, \tau) \to (Y, \sigma) \) is called contra fbg-closed (respectively contra fbg-open) if \( f(A) \) is fbg-open (respectively fbg-closed) set in \( Y \) for each f-closed (respectively open) set A in \( X \).
Theorem 4.11 Let $f : (X, \tau) \to (Y, \sigma)$ be a map.
(a) If $f$ is contra $fb$-closed, then $f$ is $fab$-closed.
(b) If $f$ is contra $fb$-open, then $f$ is $fab$-open.

Proof. (a) Let $f(A) \leq B$, where $A$ is $f$-closed in $X$ and $B$ is $fbg$-open in $Y$. Since $f$ is contra $fb$-closed $f(A)$ is $fb$-open in $Y$. Therefore $f(A) = fbInt(f(A)) \leq fbInt(B)$. Thus $f$ is $fab$-closed.
(b) Let $A \leq f(B)$, where $B$ is $f$-open in $X$ and $A$ is $fb$-closed in $Y$. Since $f$ is contra $fb$-open $f(B)$ is $fb$-closed in $Y$. Therefore $bCl(A) \leq bCl(f(B)) = f(B)$. Thus $f$ is $fab$-closed.

The converse of the above theorem need not be true, as shown in the following example.

Example 4.12 Let $X = \{a, b, c\} = Y, \tau = \{0, 1, A\}$ and $\rho = \{0, 1, A, B\}$ where $A = \{(a, 1), (b, 0), (c, 0)\}, B = \{(a, 1), (b, 1), (c, 0)\}, f : (X, \tau) \to (Y, \rho)$ be the identity map. For $V \in X$, $f(V) \leq 1$. The image of every $f$-closed set of $X$ is contained in the only $fbg$-closed set 1 in $Y$. Then $f$ is $fab$-closed but not contra $fb$-closed.

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