Abstract

In this paper we present some properties of the generalization of the multiplicative Fibonacci sequences in case of 3F – Sequence.

Mathematics Subject Classification: 11B39, 11B37

Keywords: Fibonacci sequence, triple sequences.

Introduction

The concept of Fibonacci Triple sequence was first introduced by J.Z. Lee and J.S.Lee [1]. The Fibonacci Triple Sequence is a new direction in generalization of coupled Fibonacci sequences. Fibonacci Triple sequences generalized by K.T. Atanassov [3]. Much work has been done to study on Fibonacci triple sequences. K.T. Atanassov [2, 3] described new ideas for Fibonacci triple sequences and called 3F sequences.
We shall study on multiplicative triple Fibonacci sequences and derived some properties of second order.

1. **Multiplicative Fibonacci Triple Sequence of Second Order:** Multiplicative Fibonacci sequence is the explosive development in the region of Fibonacci sequence. Let \( \{\alpha_i\}_{i=0}^{\infty} \), \( \{\beta_i\}_{i=0}^{\infty} \) and \( \{\gamma_i\}_{i=0}^{\infty} \) be three infinite sequences and six arbitrary real numbers \( a, b, c, d, e, f \) be given. We defined following specific scheme for three sequences.

**First Scheme:**

\[
\begin{align*}
\alpha_0 &= a, \quad \beta_0 = b, \quad \gamma_0 = c \\
\alpha_1 &= d, \quad \beta_1 = e, \quad \gamma_1 = f \\
\alpha_{n+2} &= \beta_{n+1} \gamma_n, \quad n \geq 0 \quad (1) \\
\beta_{n+2} &= \gamma_{n+1} \alpha_n, \quad n \geq 0 \\
\gamma_{n+2} &= \alpha_{n+1} \beta_n, \quad n \geq 0.
\end{align*}
\]

**Second Scheme:**

\[
\begin{align*}
\alpha_0 &= a, \quad \beta_0 = b, \quad \gamma_0 = c \\
\alpha_1 &= d, \quad \beta_1 = e, \quad \gamma_1 = f \\
\alpha_{n+2} &= \gamma_{n+1} \beta_n, \quad n \geq 0 \quad (2) \\
\beta_{n+2} &= \alpha_{n+1} \gamma_n, \quad n \geq 0 \\
\gamma_{n+2} &= \beta_{n+1} \alpha_n, \quad n \geq 0.
\end{align*}
\]

There are 36 different schemes of 3F sequences. There are ten schemes are trivial 3F sequences because they having at least one resulting sequence same as Fibonacci sequence.

2. **Properties of first Scheme**

First few terms of first scheme are as under:
Fibonacci triple sequences

If we set $a = b = c$ and $d = e = f$, then the sequences will coincide with each other and with the sequences $\{F_i\}$ which is called a generalized Fibonacci sequence.

where

$$
F_0(a,d) = a, \quad F_1(a,d) = d,
$$

and

$$
F_{n+2}(a,d) = F_{n+1}(a,d) \cdot F_n(a,d).
$$

Now we present fundamental properties of first scheme.

**Theorem : 2.1** For every integer $n \geq 0$,

\begin{align*}
(a) \quad \prod_{k=0}^{n} \gamma_{4k+5} \cdot \alpha_{4k+4} &= \prod_{k=0}^{n} \beta_{4k+6} \\
(b) \quad \prod_{k=0}^{n} \alpha_{4k+5} \cdot \beta_{4k+4} &= \prod_{k=0}^{n} \gamma_{4k+6} \\
(c) \quad \prod_{k=0}^{n} \beta_{4k+5} \cdot \gamma_{4k+4} &= \prod_{k=0}^{n} \alpha_{4k+6}
\end{align*}

To prove this we shall use induction method.

**Proof (a)**

\[
\prod_{k=0}^{n} \gamma_{4k+5} \cdot \alpha_{4k+4} = \prod_{k=0}^{n} \beta_{4k+6} \quad \text{for } n \geq 0
\]

If $n = 0$, the result is true because

\[
\gamma_5 \cdot \alpha_4 = \beta_6 = b^3 \cdot d^5 \cdot b^2 \cdot d^3 = b^5 \cdot d^8
\]
= β₆
i.e., γ₅·α₄ = β₆

Hence the result is true for \( n \geq 0 \). Similarly proof can be given for remaining parts (b) and (c).

**Theorem : 2.2.** For any integer \( n \geq 0 \),
(a) \( \alpha_{n+7} = (\alpha_{n+4})^4·\alpha_{n+1} \)
(b) \( \beta_{n+7} = (\beta_{n+4})^4·\beta_{n+1} \)
(c) \( \gamma_{n+7} = (\gamma_{n+4})^4·\gamma_{n+1} \)

**Proof : (a)** To prove this we shall use induction method.

If \( n = 0 \) then
\[
\alpha_7 = \beta_6·\gamma_5 \quad \text{(by first scheme)}
\]
\[
= (\gamma_5·\alpha_4)·\gamma_5 \quad \text{(by first scheme)}
\]
\[
= (\gamma_5)^2·\alpha_4 \quad \text{(by first scheme)}
\]
\[
= (\alpha_4·\beta_3)^2·\alpha_4
\]
\[
= (\alpha_4)^3·\beta_3·\gamma_2·\alpha_1
\]
\[
= (\alpha_4)^3·(\beta_3·\gamma_2)·\alpha_1
\]
\[
= (\alpha_4)^3·(\beta_3·\gamma_2)·\alpha_1
\]
\[
= (\alpha_4)^4·\alpha_1
\]

Thus the result is true for \( n = 0 \).

Let us assume that the result is true for some integer \( n \geq 1 \). Then
\[
\alpha_{n+8} = \beta_{n+7}·\gamma_{n+6} \quad \text{(by first scheme)}
\]
\[
= \gamma_{n+6}·\alpha_{n+5}·\alpha_{n+5}·\beta_{n+4} \quad \text{(by first scheme)}
\]
\[
= \gamma_{n+6}·(\alpha_{n+5})^2·\beta_{n+4}
\]
\[
= (\alpha_{n+5})^3·\beta_{n+4}·(\alpha_{n+5})^5·\beta_{n+4} \quad \text{(by first scheme)}
\]
\[
= (\alpha_{n+5})^3·(\beta_{n+4})^2
\]
\[
= (\alpha_{n+5})^3·(\gamma_{n+3}·\alpha_{n+2})^2
\]
\[
= (\alpha_{n+5})^3·(\gamma_{n+3}·\alpha_{n+2})·\beta_{n+1}·(\alpha_{n+2})^2
\]
\[
= (\alpha_{n+5})^3·\beta_{n+4}·(\beta_{n+1}·\alpha_{n+2})·(\alpha_{n+2})
\]
\[
= (\alpha_{n+5})^3·(\beta_{n+4}·\gamma_{n+3})·\alpha_{n+2}
\]
\[
= (\alpha_{n+5})^3·(\gamma_{n+3}·\alpha_{n+2})·\beta_{n+1}·(\alpha_{n+2})
\]
\[
= (\alpha_{n+5})^3·\alpha_{n+2}
\]

i.e, \( \alpha_{n+8} = (\alpha_{n+5})^4·\alpha_{n+2} \)

Hence, the result is true for all integers \( n \geq 0 \). Similar proof can be given for remaining parts (b) and (c).
3. Properties of second scheme

Second scheme:
\[\alpha_0 = a, \beta_0 = b, \gamma_0 = c\]
\[\alpha_1 = d, \beta_1 = e, \gamma_1 = f\]
\[\alpha_{n+2} = \gamma_{n+1} \cdot \beta_n, \ n \geq 0\]
\[\beta_{n+2} = \alpha_{n+1} \cdot \gamma_n, \ n \geq 0\]
\[\gamma_{n+2} = \beta_{n+1} \cdot \alpha_n, \ n \geq 0\]

First few terms of second scheme are as under:

<table>
<thead>
<tr>
<th>n</th>
<th>(\alpha_n)</th>
<th>(\beta_n)</th>
<th>(\gamma_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>d</td>
<td>e</td>
<td>f</td>
</tr>
<tr>
<td>2</td>
<td>bf</td>
<td>cd</td>
<td>ae</td>
</tr>
<tr>
<td>3</td>
<td>a^2</td>
<td>bf^2</td>
<td>cd^2</td>
</tr>
<tr>
<td>4</td>
<td>c^2d^3</td>
<td>a^4e^3</td>
<td>b^5f^6</td>
</tr>
<tr>
<td>5</td>
<td>b^6f^9</td>
<td>c^5d^9</td>
<td>a^7e^9</td>
</tr>
<tr>
<td>6</td>
<td>a^8e^{13}</td>
<td>b^11f^{18}</td>
<td>c^9d^{13}</td>
</tr>
<tr>
<td>7</td>
<td>c^{13}d^{13}</td>
<td>a^{17}e^{17}</td>
<td>b^{17}f^{34}</td>
</tr>
<tr>
<td>8</td>
<td>b^{13}f^{21}</td>
<td>c^{21}d^{21}</td>
<td>a^{17}e^{31}</td>
</tr>
<tr>
<td>9</td>
<td>a^{24}e^{34}</td>
<td>b^{21}f^{34}</td>
<td>c^{31}d^{34}</td>
</tr>
<tr>
<td>10</td>
<td>c^{35}d^{35}</td>
<td>a^{34}e^{55}</td>
<td>b^{35}f^{55}</td>
</tr>
</tbody>
</table>

**Theorem 3.1**

For any integer \(n \geq 0\)

(a) \(\alpha_{n+7} = (\alpha_{n+4})^4 \cdot \alpha_{n+1}\)
(b) \(\beta_{n+7} = (\beta_{n+4})^4 \cdot \beta_{n+1}\)
(c) \(\gamma_{n+7} = (\gamma_{n+4})^4 \cdot \gamma_{n+1}\)

The theorem can be proved by Induction method.

**References**


Received: January, 2012