Helix Hypersurfaces and Special Curves

Evren ZIPLAR

Department of Mathematics, Faculty of Science
University of Ankara, Tandoğan, Ankara, Turkey
evrenziplar@yahoo.com

Ali ŞENOL

Cankırı Karatekin University, Faculty of Science
Department of Mathematics, Cankırı, Turkey

Yusuf YAYLI

Department of Mathematics, Faculty of Science
University of Ankara, Tandoğan, Ankara, Turkey

Abstract

In this paper, we study special curves on helix hypersurfaces whose tangent space makes a constant angle with a fixed direction in Euclidean $\mathbb{E}^n$. Besides, we investigate special curves on $r$-helix hypersurfaces whose tangent space makes a constant angle with $r$ linearly independent directions. Also, we give the relations between helix hypersurfaces and the Gauss transformations of these surfaces in Euclidean $n$-space.

Mathematics Subject Classification: 53A04, 53B25, 53C40, 53C50

Keywords: $r$-helix hypersurface, Gauss transformation, line of curvature, geodesic curve, asymptotic curve, slant helix

1. Introduction

In differential geometry of surfaces, an helix hypersurface in $\mathbb{E}^n$ is defined
by the property that tangent planes make a constant angle with a fixed direction in
[2]. Di Scala and Ruiz- Hernández have introduced the concept of these surfaces
in [2]. Moreover, they have studied $r$-helices submanifolds [2]. That is to say
submanifolds such that its tangent space makes a constant angle with $r$ linearly
independent directions.

Helix hypersurfaces has been worked in nonflat ambient spaces in [3,4].
Cermelli and Di Scala have also studied helix hypersurfaces in liquid cristals in
[8].

Nowadays, M. Ghomi worked out the shadow problem given by H.Wente.
And, He mentioned the shadow boundary in [7]. Ruiz- Hernández investigated
that shadow bounderies are related to helix submanifolds whose tangent space
makes constant angle with a fixed direction in [5].

A.I. Nistor has also introduced certain constant angle surfaces constructed on
curves in $\mathbb{E}^3$ in [1]. Özkaldi and Yayloley give some characterization for a curve
lying on a surface for which the unit normal makes a constant angle with a fixed
direction in [9].

One of the main purposes of this work is to observe the relations between helix
hypersurfaces and special curves in Euclidean $n$-space $\mathbb{E}^n$. Another purpose of
this study is to give the relations between helix hypersurfaces and the Gauss
transformations of these surfaces in Euclidean $n$-space $\mathbb{E}^n$.

2. Preliminaries

**Definition 2.1** Let $\alpha: \mathbb{R} \rightarrow E^n$ be an arbitrary curve in $E^n$. Recall that
the curve $\alpha$ is said to be of unit speed (or parametrized by the arc-length
function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot,\cdot \rangle$ is the standard scalar product in the
Euclidean space $E^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i,$$

for each $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in E^n$.

Let $\{V_1(s), V_2(s), \ldots, V_n(s)\}$ be the moving frame along $\alpha$, where the vectors $V_i$
are mutually orthogonal vectors satisfying $\langle V_i, V_j \rangle = 1$. The Frenet equations for
$\alpha$ are given by

$$
\begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3 \\
\vdots \\
V'_{n-1} \\
V'_n
\end{bmatrix}
=
\begin{bmatrix}
0 & k_1 & 0 & 0 & \cdots & 0 & 0 \\
-k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 \\
0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & k_{n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & -k_{n-1} & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{n-1} \\
V_n
\end{bmatrix}.
$$
Recall that the functions $k_i(s)$ are called the $i$-th curvatures of $\alpha$ [6].

**Definition 2.2** Let $\alpha : I \subset \mathbb{R} \to E^n$ be a unit speed curve with nonzero curvatures $k_i$ ($i = 1, 2, \ldots, n$) in $E^n$ and let $\{V_1, V_2, \ldots, V_n\}$ denote the Frenet frame of the curve $\alpha$. We call $\alpha$ a $V_n$-slant helix, if the $n$-th unit vector field $V_n$ makes a constant angle $\varphi$ with a fixed direction $X$, that is,

$$\langle V_n, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant}$$

along the curve, where $X$ is unit vector field in $E^n$ [6].

**Definition 2.3** Let $\alpha : I \subset \mathbb{R} \to M$ be a unit speed curve on a hypersurface $M$ in Euclidean 3-space $E^3$. Take the frame $\{T, Y, N\}$ along the curve $\alpha$, where $T$ is the tangent vector field of $\alpha$ and $N$ is a normal vector field on $M$. Then, the frame $\{T, Y, N\}$ is called Darboux frame of $\alpha$ with respect to $M$ and the vector fields $T, Y, N$ are called Darboux vector fields of $\alpha$ with respect to $M$.

The Darboux equations for this frame are

$$T' = k_g Y + k_n N,$$

$$Y' = -k_g T + \tau_g N,$$

$$N' = -k_n T - \tau_g Y.$$  

Here $k_n$ is called the normal curvature of $\alpha$ on $M$, $k_g$ is called the geodesic curvature of $\alpha$ on $M$ and $\tau_g$ is called the geodesic torsion of $\alpha$ on $M$ [9].

**Definition 2.4** Given a hypersurface $M \subset \mathbb{R}^n$ and an unitary vector $d \neq 0$ in $\mathbb{R}^n$, we say that $M$ is a helix with respect to the fixed direction $d$ if for each $q \in M$ the angle between $d$ and $T_q M$ is constant. Note that the above definition is equivalent to the fact that $\langle d, \xi \rangle$ is constant function along $M$, where $\xi$ is a normal vector field on $M$ [2].

**Theorem 2.1** Let $H \subset \mathbb{R}^{n-1}$ be an orientable hypersurface in $\mathbb{R}^{n-1}$ and let $\eta$ be an unitary normal vector field of $H$. Then,

$$f_\theta(x, s) = x + s(\sin(\theta)\eta(x) + \cos(\theta)d), f_\theta : H \times \mathbb{R} \to \mathbb{R}^n, \theta = \text{constant}$$

is a helix with respect to the fixed direction $d$ in $\mathbb{R}^n$, where $x \in H$ and $s \in \mathbb{R}$. Here $d$ is the vector $(0, 0, \ldots, 1) \in \mathbb{R}^n$ such that $d$ is orthogonal to $\eta$ and $H$.

Besides, $\xi(x) = -\cos(\theta)\eta(x) + \sin(\theta)d$ is an unitary normal vector field of $f_\theta$, where $x \in H$ [2].

**Definition 2.5** A submanifold $M \subset \mathbb{R}^n$ is a $r$-helix if there exist a linear subspace $H \subset \mathbb{R}^n$ of dimension $r = \dim(H)$ such that $M$ is a helix with
respect to any direction \( d \in H \). The subspace \( H \) is called the subspace of helix directions [2].

3. MAIN THEOREMS

**Theorem 3.1** Let \( M \) be a helix hypersurface with the direction \( d \) in \( E^3 \) and let \( \alpha : I \subset \mathbb{R} \to M \) be a unit speed curve on \( M \). If \( d \in Sp\{N,Y\} \) along the curve \( \alpha \), then \( \alpha \) is a line of curvature on \( M \) and \( \frac{k_s}{k_g} = -\tan(\theta) = \text{constant} \), where \( \{N,Y\} \) are Darboux vector fields of \( \alpha \) and \( \theta \) is the constant angle between \( d \) and \( N \).

Conversely, if \( \alpha \) is a line of curvature on \( M \), then \( d \in Sp\{N,Y\} \).

**Proof:** Since \( M \) is a helix hypersurface with the direction \( d \) and \( d \in Sp\{N,Y\} \), we can write as
\[
d = \cos(\theta)N + \sin(\theta)Y
\] along the curve \( \alpha \), where \( <N,d> = \cos(\theta) = \text{constant} \). Taking the derivative in each part of the equation (1), we obtain:
\[
0 = \cos(\theta)N' + \sin(\theta)Y'.
\]
Using the Darboux equations in (*), we have
\[
\left(-k_n \cos(\theta) - k_g \sin(\theta)\right)T' + \left(-\tau_g \cos(\theta)\right)Y' + \tau_g \sin(\theta)N = 0
\] (2)
Since the system \( \{T,Y,N\} \) is linear independent, we deduce from the equation (2) that
\[
k_n \cos(\theta) + k_g \sin(\theta) = 0
\] (3)
\[
\tau_g \cos(\theta) = 0
\] (4)
\[
\tau_g \sin(\theta) = 0
\] (5)
So, from the equations (4) and (5), we get \( \tau_g = 0 \). It follows that the curve \( \alpha \) is a line of curvature on \( M \). Moreover, from the equation (3), we have \( \frac{k_s}{k_g} = -\tan(\theta) = \text{constant} \).

Conversely, we assume that \( \alpha \) is a line of curvature on \( M \). Then, it follows that
\[
\tau_g = 0
\] (6)
On the other hand,
\[
< N, d > = \text{constant}
\] (7)
By taking the derivative of the equation (7), we obtain
where $N' = -k_n T - \tau_g Y$ . And, so we have $N' = -k_n T$ by using (6). Consequently, if we also consider (8), we obtain $\langle T, d \rangle = 0$ . It follows that $d \in Sp\{N, Y\}$.

This completes the proof.

**Theorem 3.2** Let $M$ be a helix hypersurface with the direction $d$ in $E^3$ and let $\alpha : I \subset IR \to M$ be a unit speed curve on $M$ . If $d \in Sp\{N, T\}$ along the curve $\alpha$ , then $\alpha$ is a asymptotic curve on $M$ and $\frac{\tau_g}{k_g} = \tan(\theta) = \text{constant}$, where $\{N, T\}$ are Darboux vector fields of $\alpha$ and $\theta$ is the constant angle between $d$ and $N$ .

Conversely, if $\alpha$ is a asymptotic curve on $M$ , then $d \in Sp\{N, T\}$.

**Proof:** Since $M$ is a helix hypersurface with the direction $d$ and $d \in Sp\{N, T\}$, we can write as

$$d = \cos(\theta)N + \sin(\theta)T$$

along the curve $\alpha$ , where $\langle N, d \rangle = \cos(\theta) = \text{constant}$ . Taking the derivative in each part of the equation (9), we obtain:

$$0 = \cos(\theta)N' + \sin(\theta)T' .$$

Using the Darboux equations in ( * ), we have

$$(-k_n \cos(\theta)T) + (-\tau_g \cos(\theta) + k_g \sin(\theta))Y + (k_n \sin(\theta))N = 0$$

Since the system $\{T, Y, N\}$ is linear independent, we deduce from the equation (10) that

$$-\tau_g \cos(\theta) + k_g \sin(\theta) = 0$$

$$k_n \cos(\theta) = 0$$

$$k_n \sin(\theta) = 0$$

So, from the equations (12) and (13), we get $k_n = 0$ . It follows that the curve $\alpha$ is a asymptotic curve on $M$ . Moreover, from the equation (11), we have $\frac{\tau_g}{k_g} = \tan(\theta) = \text{cnst}$.

Conversely, we assume that $\alpha$ is a asymptotic curve on $M$ . Then, it follows that

$$k_n = 0$$

On the other hand,

$$\langle N, d \rangle = \text{constant}$$
By taking the derivative of the equation (15), we obtain
\[ <N', d> = 0 \]  
where \( N' = -k_n T - \tau_g Y \). And, so we have \( N' = -\tau_g Y \) by using (14). Consequently, if we also consider (16), we obtain \( <Y, d> = 0 \). It follows that \( d \in Sp[N, T] \).

This completes the proof.

**Theorem 3.3** Let \( M \) be a \( r \)-helix hypersurface in \( E^n \) and let \( H \) be the subspace of helix directions of \( M \). If \( \alpha : I \subset IR \rightarrow M \) be a unit speed geodesic curve on \( M \), then the curve \( \alpha \) is a \( V_2 \)-slant helix with respect to any direction \( d \in H \) in \( E^n \).

**Proof:** Let \( \xi \) be a normal vector field on \( M \). Since \( M \) is \( r \)-helix hypersurface, \( <d, \xi> = \) constant, where \( d \in H \) is any direction. That is, the angle between \( d \) and \( \xi \) is constant on every point of the surface \( M \). And, \( \alpha'(s) = \lambda \xi |_{\alpha(s)} \) along the curve \( \alpha \) since \( \alpha \) is a geodesic curve on \( M \). Moreover, by using the Frenet equation \( \alpha'(s) = V_1 = k_1 V_2 \), we obtain \( \lambda \xi |_{\alpha(s)} = k_1 V_2 \), where \( k_1 \) is the first curvature of \( \alpha \). Thus, from the last equation, by taking norms on both sides, we obtain \( \xi = V_2 \) or \( \xi = -V_2 \). So, \( \langle d, V_2 \rangle \) is constant along the curve \( \alpha \) since \( \langle d, \xi \rangle = \) constant. In other words, the angle between \( d \) and \( V_2 \) is constant along the curve \( \alpha \). Consequently, the curve \( \alpha \) is a \( V_2 \)-slant helix with respect to any direction \( d \in H \) in \( E^n \).

This latter Theorem has the following corollary.

**Corollary 3.1** Let \( M \) be a helix hypersurface with the direction \( d \) in \( E^n \) and let \( \alpha : I \subset IR \rightarrow M \) be a unit speed geodesic curve on \( M \). Then, the curve \( \alpha \) is a \( V_2 \)-slant helix with the direction \( d \) in \( E^n \) [10].

**Theorem 3.4** Let \( M \) be a \( r \)-helix hypersurface in \( E^n \) and let \( H \) be the subspace of helix directions of \( M \). If \( \alpha : I \subset IR \rightarrow M \) a unit speed curve on \( M \) and the \( n \)-th unit vector field \( V_n \) of \( \alpha \) equals to \( \xi \) or \( -\xi \), then \( \alpha \) is a \( V_n \)-slant helix with respect to any direction \( d \in H \) in \( E^n \), where \( \xi \) is a normal vector field on \( M \).

**Proof:** Let \( \xi \) be a normal vector field on \( M \). Since \( M \) is \( r \)-helix hypersurface, \( <d, \xi> = \) constant, where \( d \in H \) is any direction. That is, the angle between \( d \) and \( \xi \) is constant on every point of the surface \( M \). Let the \( n \)-th unit vector field \( V_n \) of \( \alpha \) be equals to \( \xi \) or \( -\xi \). Then \( \langle d, V_n \rangle \) is constant along the curve \( \alpha \) since \( \langle d, \xi \rangle = \) constant. That is, the angle between \( d \) and \( V_n \) is constant along the curve \( \alpha \). Finally, the curve \( \alpha \) is a \( V_n \)-slant helix with respect to any direction \( d \in H \) in \( E^n \).
Theorem 3.4 has the following corollary.

**Corollary 3.2** Let $M$ be a helix hypersurface in $E^n$ and let $\alpha : I \subset \mathbb{R} \to M$ be a unit speed curve on $M$. If the $n$-th unit vector field $V_n$ of $\alpha$ equals to $\xi$ or $-\xi$, then $\alpha$ is a $V_n$-slant helix in $E^n$, where $\xi$ is a normal vector field on $M$ [10].

**Theorem 3.5** Let $M$ be a $r-$helix hypersurface in $E^n$ and let $H$ be the subspace of helix directions of $M$. If $\alpha : I \subset \mathbb{R} \to M (\alpha(t) \in M, t \in I)$ is a curve on the surface $M$ and $\alpha$ is a line of curvature on $M$, then $d \in Sp\{T\}^\perp$ along the curve $\alpha$, where $T$ is tangent vector field of $\alpha$ and $d \in H$ is any direction.

**Proof:** Since $M$ is a $r-$helix hypersurface, 

$$<N, d> = \text{constant} \ (d \in H \text{ is any direction})$$

along the curve $\alpha$, where $N$ is the normal vector field of $M$. If we are taking the derivative in each part of the equality with respect to $t$, we obtain:

$$\langle (N)' , d \rangle = 0.$$

along the curve $\alpha$. Since $\alpha$ is a line of curvature on $M$, $(N)' = S(T) = \lambda T$ along the curve $\alpha$, where $S$ is the shape operator of the surface $M$. So, we have

$$\langle T, d \rangle = 0.$$

Finally, $d \in Sp\{T\}^\perp$ along the curve $\alpha$, where $d \in H$ is any direction.

The latter Theorem has the following corollary.

**Corollary 3.3** Let $M$ be a helix hypersurface with the direction $d$ in $E^n$ and let $\alpha : I \subset \mathbb{R} \to M (\alpha(t) \in M, t \in I)$ be a curve on the surface $M$. If $\alpha$ is a line of curvature on $M$, then $d \in Sp\{T\}^\perp$ along the curve $\alpha$, where $T$ is tangent vector field of $\alpha$ [10].

**Theorem 3.6** Let $M$ be a hypersurface in Euclidean $n$-space $E^n$ and let $N$ be the unit normal vector field of $M$. If $M$ is a 1-helix in $E^n$, then the position vectors on all points of the surface

$$\eta(M) = \{X \in S^{n-1} \mid X = N(P), P \in M\}$$

make a constant angle with a fixed direction in $E^n$. Here,

$$\eta : M \to S^{n-1} \subset E^n$$

$$P \to \eta(P) = \tilde{N}(P)$$

is Gauss transformation of $M$, where $S^{n-1}$ is the unit hypersphere in $E^n$.

**Proof:** Let $d \in \mathbb{R}^n$ be the helix direction of $M$. Since $M$ is a 1-helix in $E^n$,
\[ < N(P), d >= \cos(\theta) = \text{constant} \]

for every \( P \in M \). So, we obtain

\[ < X, d >= \cos(\theta) = \text{constant} , \]

for every \( X = \tilde{N}(P) \in \eta(M) \). At the same time, since \( X = \tilde{N}(P) \) is the position vector of \( \eta(M) \) on the unit hypersphere \( S^n \), \( \eta(M) \) make a constant angle with a fixed direction in \( E^n \). This completes the proof.

The following examples 3.1 and 3.2 are related to the Theorem 3.6.

**Example 3.1** We consider the hypersurface \( M \) given by

\[
\phi: M \rightarrow E^3, \quad \phi(u, v) = \left( \cos u - \left(\frac{\sqrt{2}}{2}\right)\cos u, \sin u - \left(\frac{\sqrt{2}}{2}\right)\cos u, \frac{\sqrt{2}}{2} \right).
\]

Let us determine the normal to the surface. To do this, we compute the partial derivatives of \( \phi \) with respect to \( u \) and \( v \).

\[
\phi_u = \left( -\sin u + \left(\frac{\sqrt{2}}{2}\right)\sin u, \cos u + \left(\frac{\sqrt{2}}{2}\right)\sin u, 0 \right), \quad \phi_v = \left( -\frac{\sqrt{2}}{2}\cos u, -\frac{\sqrt{2}}{2}\cos u, \frac{\sqrt{2}}{2} \right).
\]

Using by \( \phi_u \) and \( \phi_v \), the normal to the surface is given by

\[
N = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|} = \left( \frac{\sqrt{2}}{2}\cos u, \frac{\sqrt{2}}{2}\sin u, \frac{\sqrt{2}}{2} \right).
\]

And, \( < N, d >= \frac{\sqrt{2}}{2} = \text{constant} \), where \( d = (0,0,1) \in E^3 \). That is, \( \phi \) is a 1-helix with respect to \( d \). So, the range of \( \eta: M \rightarrow \mathbb{E} \) is shown in Figure 1 and the range of the Gauss transformation of \( M \) is shown in Figure 2.
Example 3.2 In Theorem 2.1, let us take the hypersurface \( H \subset \mathbb{E}^3 \) as 
\[
\phi : H \to \mathbb{E}^3, \quad \phi(u,v) = (u, v, u.v).
\]

Let us determine the normal to the surface \( H \). To do this, we compute the partial derivatives of \( \phi \) with respect to \( u \) and \( v \).
\[
\phi_u = (1,0,v), \quad \phi_v = (0,1,u).
\]

Using by \( \phi_u \) and \( \phi_v \), the normal to the surface \( H \) is given by
\[
\eta = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|} = \frac{1}{\sqrt{v^2 + u^2 + 1}} (-v,-u,1).
\]

So, according to Theorem 2.1, the hypersurface \( M = f_{\phi} \) given by
\[
\varphi : M \to \mathbb{E}^4, \varphi(u,v,s) = (u,v,u.v,0) + s\left(\frac{\sqrt{2}}{2} \frac{1}{\sqrt{v^2 + u^2 + 1}} (-v,-u,1,0) + \frac{\sqrt{2}}{2} (0,0,0,1)\right)
\]
is a 1-helix in \( \mathbb{E}^4 \) with respect to \( d = (0,0,0,1) \in \mathbb{E}^4 \), where \( \theta = \pi/4 \) and \( s \in \mathbb{R} \).

Also, according to Theorem 2.1, the normal vector field of \( M \) :
\[
\xi = -\frac{\sqrt{2}}{2} \frac{1}{\sqrt{v^2 + u^2 + 1}} (-v,-u,1,0) + \frac{\sqrt{2}}{2} (0,0,0,1).
\]

Hence, the range of \( \mu : \)
\[
\mu(M) = \frac{\sqrt{2}}{2} \left( \frac{v}{\sqrt{v^2+u^2+1}}, \frac{u}{\sqrt{v^2+u^2+1}}, \frac{-1}{\sqrt{v^2+u^2+1}} \right),
\]

where \( \mu : M \rightarrow S^3 \subset E^4 \) is the Gauss transformation of \( M \).

Finally, \( \mu(M) \) makes a constant angle with the fixed direction \( d \in E^4 \) on \( S^3 \subset E^4 \) if we consider the Theorem 3.6.

**Theorem 3.7** Let \( M \) be a 1-helix hypersurface with the direction \( d \) in Euclidean \( n \)-space \( E^n \) and let \( d \in T_pM \) be for every \( p \in M \). If \( \alpha : I \subset IR \rightarrow M \) \((\alpha(t) \in I, t \in I)\) is a geodesic curve on the surface \( M \), then the curve

\[
\eta(\alpha(t)) = \{X \in S^{n-1} \mid X = N|_{\alpha(t)}, \alpha(t) \in M\}
\]

can not be a geodesic curve on the unit hypersphere \( S^{n-1} \subset E^n \), where \( N|_{\alpha(t)} \) the unit normal vector field of \( M \) along the curve \( \alpha \) and

\[
\eta : M \rightarrow S^{n-1} \subset E^n
\]

\[
\alpha(t) \rightarrow \eta(\alpha(t)) = \tilde{N}|_{\alpha(t)}
\]

Gauss transformation of \( M \) on the curve \( \alpha \).

**Proof:** Since \( \alpha : I \subset IR \rightarrow M \) is a geodesic curve on the surface \( M \), we can write as

\[
\alpha''(t) = \lambda N|_{\alpha(t)}
\]

along the curve \( \alpha \). We assume that \( \beta = \eta(\alpha(t)) \), where \( \eta \) Gauss transformation of \( M \). So, we obtain

\[
\beta = \frac{1}{\lambda} \alpha''
\]

since \( \alpha''(t) = \lambda N|_{\alpha(t)} \) and \( \eta(\alpha(t)) = N|_{\alpha(t)} \) along the curve \( \alpha \). On the other hand,

\[
<N|_{\alpha(t)}, d> = \cos(\theta) = \text{constant}
\]

along the curve \( \alpha \) since \( M \) is a 1-helix surface with the direction \( d \). Thus, we have

\[
<N|_{\alpha(t)}, d> = \frac{1}{\lambda} \alpha''(t), d>
\]

\[= \beta(t), d >
\]

\[= \cos(\theta)
\]

\[= \text{constant}.
\]

If we take the derivative in each part of the equality \( \beta(t), d > = \cos(\theta) \) twice, we obtain:

\[
\beta'', d > = 0.
\]

(17)
Now, we suppose that the curve $\beta$ is a geodesic curve on $S^{n-1}$. Then $\beta^n = kN|_{a(t)}$, where $N|_{a(t)}$ is also the normal vector field of $S^{n-1}$. Hence, we get

$$< N|_{a(t)}, d > = 0$$

by using the equation (17). It follows that $d \in T_{a(t)}M$ for every point $a(t) \in M$. But, according to the hypothesis in this Theorem, $d \notin T_pM$ for every $p \in M$. So, it is a contradiction. As a result, $\beta = \eta(a(t))$ can not be a geodesic curve on the unit hypersphere $S^{n-1} \subset E^n$.

**Theorem 3.8** Let $M$ be a 1-helix hypersurface with the direction $d$ in Euclidean $n$-space $E^n$ and let $\alpha: 1 \subset IR \rightarrow M$ be a curve on $M$. If the system $\{T_1, T\}$ is linear dependent along the curve $\alpha$, where $T_1$ the tangent component of $d$ and $T$ the tangent to the curve $\alpha$, then $\alpha$ is a asymptotic curve on $M$.

**Proof:** Since $M$ is a 1-helix hypersurface with the direction $d$, $< N, d > = \cos(\theta) = \text{constant}$, where $N$ is the unit normal vector field of $M$ along the curve $\alpha$. And, we can decompose the vector $d$ in its normal and tangent components:

$$d = \cos(\theta)N + \sin(\theta)T_1,$$  \hspace{1cm} (18)

where $\theta$ is constant. If we take derivative in each part of equality $< N, d > = \cos(\theta)$, we obtain $< N', d > = 0$. By using the equation (18), we can write:

$$0 = < N', d > = < N', \cos(\theta)N + \sin(\theta)T_1 >$$

$$= \cos(\theta) < N', N > + \sin(\theta) < N', T_1 > .$$

On the other hand, $< N', N > = 0$ since $N$ is the unit normal vector field. Therefore, we have:

$$< N', T_1 > = 0,$$

where $\theta \neq 0$. According to the this Theorem, since the system $\{T_1, T\}$ is linear dependent, we can write $T_1 = \lambda T$. Hence, we obtain:

$$< N', T > = 0.$$

Consequently, $\alpha$ is a asymptotic curve on $M$.

**Theorem 3.9** Let $M$ be a 1-helix hypersurface with the direction $d$ in Euclidean $n$-space $E^n$ and let $\alpha: 1 \subset IR \rightarrow M$ be a curve on $M$. If $\alpha$ is a line of curvature on $M$, then $d \in \text{Sp}[T_2, T_3, ..., T_{n-1}, N]$ , where $\text{Sp}[T, T_2, T_3, ..., T_{n-1}, N] = \chi(E^n)$ and $(T_2, T_3, ..., T_{n-1})$ vector fields in $\chi(E^n)$. Note that $T$ the unit tangent vector field of $\alpha$ and $N$ the unit normal vector field of $M$.
Proof: Since $M$ is a 1-helix hypersurface with the direction $d$, $\langle N,d \rangle = \cos(\theta) =$ constant, where $N$ is the unit normal vector field of $M$ along the curve $\alpha$. If we take derivative in each part of equality $\langle N,d \rangle = \cos(\theta)$, we obtain $\langle N',d \rangle = 0$. Moreover, $N' = \lambda T$ since $\alpha$ is a line curvature. Hence, we have: $\langle T,d \rangle = 0$.

It follows that $d \in Sp\{T_2,T_3,\ldots,T_{n-1},N\}$. This completes the proof.

Theorem 3.10 Let $M$ be a 1-helix hypersurface with the direction $d$ in Euclidean $n$-space $E^n$ and let $\alpha:1 \subset \mathbb{R} \rightarrow M$ be a curve on $M$. If the system $\{T_1',T\}$ is linear dependent, where $T_1'$ the derivative of the tangent component of $d$ and $T$ the tangent to the curve $\alpha$, then $\alpha$ is a line of curvature on $M$.

Proof: Since $M$ is a 1-helix hypersurface with the direction $d$, we can decompose the vector $d$ in its normal and tangent components:

$$d = \cos(\theta)N + \sin(\theta)T_1,$$ (19)

where $\langle N,d \rangle = \cos(\theta) =$ constant and $N$ the unit normal vector field of $M$. If we take derivative in each part of the equation (19), we obtain:

$$0 = \sin(\theta)T_1' + \cos(\theta)N'.$$ (20)

According to the hypothesis, the system $\{T_1',T\}$ is linear dependent. Hence, we can write $T_1' = \lambda T$. By using the equation (20), we have the system $\{N',T\}$ is linear dependent. It follows that $\alpha$ is a line of curvature on $M$. This completes the proof.

REFERENCES


Received: November, 2011