Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 31, 1519 - 1527

On Weakly Primary Subtractive Ideals over Noncommutative Semirings

Arwa Ashour

Department of Mathematics, The Islamic University of Gaza
Gaza, Palestine, P.O. Box 108
arashour@iugaza.edu.ps

Samia Saeed Alazab AbedRabou

Department of Mathematics, College of Girls, Ain Shams University
Cairo, Egypt, P.O. Box 11566
info@womenfacultyasu.com

Mohammad Hamoda

Department of Mathematics, Al-Aqsa University
Gaza, Palestine, P.O. Box 4015
mamh_73@hotmail.com

Abstract

In this paper, we study the concept of weakly primary subtractive ideals over arbitrary semirings. We extend some results of [3] to noncommutative semirings with $1 \neq 0$. Some properties of weakly primary subtractive ideals over noncommutative semirings are also studied. Also we study the weakly primary ideals over noncommutative rings.

Mathematics Subject Classification: Primary 16D25, Secondary 16D80, 16N60

Keywords: Subtractive ideal, primary ideal, weakly primary ideal, completely weakly prime ideal

1 Introduction

Recently, extensive research has been done on prime and primary ideals and submodules. Weakly prime ideals in a commutative ring with non-zero identity have been introduce and studied by D.D.Anderson and E.Smith in [1]. Also weakly primary ideals in a commutative ring with non-zero identity have been introduce and studied in [3]. The primary compactly packed modules over commutative rings have been studied by A.Ashour in [2]. The structure of weakly prime ideals over noncommutative semirings with non-zero identities have been discussed by Vishnu Gupta and J.N. Chaudhari [7]. They proved that if I is a subtractive ideal of a semiring R, then I is weakly prime ideal of R iff for left ideals A and B of R, $0 \neq AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. The concept of weakly prime ideals over noncommutative rings have been studied by Y.Hirano, E.Poon and H.Tsutsui in [8]. They investigated the structure of rings, not necessarily commutative nor with identity, in which all ideals are weakly prime. This paper is concerned with generalizing some results over commutative rings to noncommutative rings.

Throughout this paper, all semirings and rings are assumed to be associative with non-zero identities.

2 Preliminary Notes

We first recall the following definitions, see [6], [7].

Definition 2.1 [6] *A non empty set R together with two associative binary operations, called addition and multiplication (denoted by + and . respectively,) is called a semiring provided:*
(i) Addition is a commutative operation and that the multiplication is distributive with respect to addition both from either side.
(ii) There exists $0 \in R$ such that $r + 0 = 0 + r = r$ and $r0 = 0r = 0$ for each $r \in R$.
In other words, semirings are just rings without subtraction.

Definition 2.2 [7] *A proper ideal I of a semiring R is called subtractive if $a,\ a + b \in I,\ b \in R$, then $b \in I$.*

Definition 2.3 [7] *A proper ideal I of a semiring R is called prime if $aRb \subseteq I$ where $a,\ b \in R$, then $a \in I$ or $b \in I$.*

V.Gupta and Chaudhari J.N, in [7] defined the weakly prime ideal over a semiring as follows:

Definition 2.4 [7] *A proper ideal of a semiring R is called weakly prime if $0 \neq aRb \subseteq I$, where $a,\ b \in R$, then $a \in I$ or $b \in I$.*

Note that in the above discussion if I is a left ideal, then we call a left prime or a left weakly prime, similarly for right ideals. If the ideal is a left prime ideal and a right prime ideal, then it is called a prime ideal, similarly for left weakly prime and right weakly prime ideals.
Now we introduce the following definitions of *Radical*, and *Nilradical* :

Definition 2.5 *Let R be a semiring and let I be a two sided ideal of R. The union of all ideals B such that $B^n \subseteq I$ for some positive integer n is a two sided ideal of R and is called the radical of I which we shall denote by $N(I)$.*

Definition 2.6 *Let R be a semiring and let I be a two sided ideal of R. The set of all elements $x \in R$ such that $x^n \in I$ for some positive integer n is said to be the $nil-radical$ of I which we shall denoted by $P(I)$.*

If I is 0 in the previous definitions we use the symbols N and P for the radicals ($rad.$ *and* $nil - rad.$) of 0.
From the above preliminary discussion and definitions, we introduce the following definition:

Definition 2.7 *A proper two sided ideal I of a semiring R is called right N primary provided $a,\ b \in R$ with $aRb \subseteq I$ implies $b \in I$ or $a \in N(I)$. The ideal I is called $left$ N primary provided $a,\ b \in R$ with $aRb \subseteq I$ implies $a \in I$ or $b \in N(I)$. The ideal I is said to be N primary provided it is both right and $left$ N primary.*

If we substitute the symbol P for N in Definition 2.7, we have the definitions of *right P primary*, *$left$ P primary* and *P primary*.
Remark: It is clear that "N prime ideal" in a semiring R is "N primary", but the converse is not true in general (Similarly for "P prime").

Notice that P is not always a two sided ideal, however If R satisfies the $A.C.C.$ for right ideals and is P primary(R is P primary semiring provided 0 is both right and left N primary ideal.), one can easily show from the proof of Theorem 2.2 of [5] and from [9] that P is a two sided ideal and $P^n = 0$ for some positive integer n. Hence in this case *P primary = N primary*.

3 Main Results

Our starting point is the following definition:

Definition 3.1 *A proper two sided ideal I of a semiring R is called right N weakly primary provided $a,\ b \in R$ with $0 \neq aRb \subseteq I$ implies $b \in I$ or $a \in N(I)$. The ideal I is called $left$ N weakly primary provided $a,\ b \in R$ with $0 \neq aRb \subseteq I$ implies $a \in I$ or $b \in N(I)$. The ideal I is called N weakly primary provided it is both right and $left$ N weakly primary.*

If in definition 3.1 we substitute the symbol P for N, we have the definitions of *right* P *weakly primary*, *left* P *weakly primary* and P *weakly primary*.
Remark: (1)It is easy to see that "N primary ideal" is "N weakly primary", but the converse is not true, because 0 is always "N weakly primary ideal" (by definition) but not necessary "N primary". So "N weakly primary ideal" does not need to be "N primary".
(Similarly for "P primary ideal").
(2) It is clear that every N weakly prime ideal in a semiring R is N weakly primary, but the converse is not true in general.
(Similarly for "P weakly primary ideal").

Lemma 3.2 *Let I be a two sided P weakly primary subtractive ideal in a semiring R. If I is not an P primary ideal, then* $I^2 = \{ab \,:\, a,\ b \in\ I\} = 0$.

Proof:
Suppose that $I^2 \neq\ 0$; we show that I is a P primary ideal of R. Let $aRb \subseteq\ I$ where $a,\ b \in\ R$. If $aRb \neq\ 0$, then $a \in\ I$ or $b \in\ P(I)$. So assume that $aRb = 0$. If $0 \neq\ aI \subseteq\ I$, then there is an element d of I such that $ad \neq\ 0$. Hence $0 \neq\ aRd = aR(d + b) \subseteq\ I$. Then either $a \in\ I$ orm $b + d \in\ P(I)$. $\Rightarrow a \in\ I$ or $b \in\ P(I)$,(because if $b + d \in\ P(I)$, then there exists a positive integer n such that
$(b+d)^n = \sum_{k=0}^{n} \binom{n}{k} b^k\, d^{n-k} \in\ I$ and since $d \in\ I$, then $b^n \in\ I$. So $b \in\ P(I)$).
Therefore I ia a P primary ideal.
Now we can assume that $aI = 0$. If $Ib \neq\ 0$, then there exists $u \in\ I$ such that $ub \neq\ 0$. Now $0 \neq\ uRb = (u + a)Rb \subseteq\ I$. So $a \in\ I$ or $b \in\ P(I)$, and hence I ia a P primary ideal. Thus we can assume that $Ib = 0$. Since $I^2 \neq\ 0$, there are elements $e, f \in\ I$ such that $ef \neq\ 0$. Then $0 \neq\ eRf = (a + e)R(b + f) \subseteq\ I$, so either $a \in\ I$ or $b \in\ P(I)$, and hence I is a P primary ideal.
Note: If we replace P by N in Lemma 3.2., then the lemma will be true if we add the following condition: R is a P primary semiring and satisfies the $A.C.C.$ for right ideals.

Theorem 3.3 *Let I be a proper two sided subtractive ideal in a semiring R. If for ideals (left or right) $A,\ B$ of R with $0 \neq\ AB \subseteq\ I$ implies $A \subseteq\ I$ or for some positive integer n, $B^n = \{b^n \in\ R : b \in\ B\} \subseteq\ I$. Then I is a P weakly primary ideal of R.*

Proof:
Suppose that I be a proper two sided subtractive ideal in a semiring R and let $0 \neq\ aRb \subseteq\ I$ where $a,\ b \in\ R$. Then $0 \neq\ \ < a >< b > \subseteq\ I$. Hence $< a > \subseteq\ I$ or $< b^n > \subseteq\ I$ for some positive integer n. So $a \in\ I$ or $b^n \in\ I$ for some positive integer n, implies $b \in\ P(I)$ and therefore I is a P weakly primary ideal of R.

Lemma 3.4 *Let I be a two sided P weakly primary subtractive ideal in a P primary semiring R that satisfies the A.C.C. for right ideals. Then $I \subseteq P$ or $P \subseteq I$ where P is the nil-radical of* 0.

Proof:
Since R is a P primary semiring that satisfies the $A.C.C.$ for right ideals, then $P = nil - radical\ of\ zero$, is a two sided ideal of R. Now if I is a P primary two sided ideal of R, then $P \subseteq I$. If I is not a P primary two sided ideal of R, then by Lemma 3.2., $I^2 = \{ab : a, b \in I\} = 0 \subseteq P$. So $I \subseteq P$.

Following [7], we have the following notations:
Let R be a semiring, $x \in R$ and let I be a two sided ideal of R such that $x \in R - N(I)$.
We define:

$$(I : Rx) = \{y \in R : yRx \subseteq I\} \ and$$

$$(I : xR) = \{y \in R : xRy \subseteq I\}.$$

They form ideals of R containing I.

Lemma 3.5 *Let R be a semiring, and let I be a two sided subtractive ideal of R. Then the following statements are equivalent:*
(1) I is an N weakly primary ideal of R.
(2) If $x \in R - N(I)$, then $(I : Rx) = I \bigcup (0 : Rx)$.
(3) If $x \in R - N(I)$, then $(I : Rx) = I$ or $(I : Rx) = (0 : Rx)$.

Proof:
(1) $\Rightarrow$ (2). Let $y \in (I : Rx)$. Then $yRx \subseteq I$. If $yRx = 0$, then $y \in (0 : Rx)$. If $yRx \neq 0$, then $y \in I$. Hence $(I : Rx) \subseteq I \bigcup (0 : Rx)$. On the other way, let $u \in I \bigcup (0 : Rx)$. Then $uRx \subseteq I$. Hence $u \in (I : Rx)$.
(2) $\Rightarrow$ (3). It follows directly by Lemma(6) and (7) in [7].
(3) $\Rightarrow$ (1). Let $yRx \subseteq I$ such that $yRx \neq 0$. Now $(I : Rx) = I$ or $(I : Rx) = (0 : Rx)$. Assume $(I : Rx) = (0 : Rx)$. As $y \in (I : Rx)$, so $yRx = 0$, a contradiction. Hence $(I : Rx) = I$ and so $y \in I$. Therefore I is an N weakly primary ideal of R.

Similarly the right analogues of Lemma 3.5. can be established.

Lemma 3.6 *Let I be a P weakly primary subtractive ideal that is not a P primary over a semiring R, then $P(I) = P$.*

Proof:
Assume that I is a P weakly primary subtractive ideal that is not a P primary over a semiring R, then it is clear that $P \subseteq P(I)$. Now by Lemma 3.2., $I^2 = 0$

gives $I \subseteq P$, hence $P(I) \subseteq P$. Therefore $P(I) = P$.

Example(1):
Let $R = Z_{12}$ and let $I = \{0, 6\}$. Then I is not an N weakly primary ideal of R $\Rightarrow I$ is not a P primary. It is a subtractive ideal.

Example(2):
Let $R = Z_{12}$ and let $I = \{0, 3, 6, 9\}$. Then I is a subtractive P primary ideal of $R \Rightarrow I$ is a P weakly primary ideal of R.

Example(3):
Let $R = (Z^+, +, .)$ and consider the semiring
$H = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\}$.

Let I be a two sided ideal of R. Then $T = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in I\}$ is a two sided ideal of H. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be in H

(1) $I =< 2, 3 > \Rightarrow 0 \neq AHB \subseteq T$, but A, B not in T and A, B not in $N(T)$. Hence T is not an N weakly primary ideal of H. It is not subtractive. Also $T^2 \neq 0$, and I is an N primary.

(2) $I =< 2 > \Rightarrow 0 \neq AHB \subseteq T$, but $A, B \notin T$ and $A, B \notin N(T)$. Hence T is not an N weakly primary ideal of H. It is subtractive. Also $T^2 \neq 0$, and I is an N primary.

Weakly Primary Ideals over Noncommutative Rings:

Now, we investigate the structure of weakly primary ideals over arbitrary rings not necessarily commutative with $1 \neq 0$. Let R be a ring. Since every ring is a semiring , so the definitions of radicals (radical and nil-radical) over a semiring are still hold over the ring R. We denote the radical of I and nil-radical of I by $N(I)$ and $P(I)$ respectively where I is a two sided ideal of the ring R. If I is 0 we use symbols N and P for the radicals of 0. Although P is not always a two sided ideal, we shall interested in P only when it is a two sided ideal. Now we recall the following definitions:

Definition 3.7 [4] *Let R be a ring. The set of all elements $x \in R$ such that $yx + 1$ is a unit of R for all $y \in R$ is a two sided ideal of R and is called the Jacobson radical of R which is denoted by J.*

From the above definition, if $x \in J$ then $xy + 1$ is also a unit of R for all $y \in R$.

Definition 3.8 [4] *A ring R is called duo provided every right ideal is a left ideal and every left ideal is a right ideal.*

One can see easily if R is a duo ring, then $Rx = xR$, $\forall\, x \in R$. Hence in this case P is a two sided ideal.
Now we introduce the following definitions.

Definition 3.9 *A proper two sided ideal I of a ring R is said to be completely weakly prime provided $0 \neq ab \in I, a \in R, b \in R$, implies $a \in I$ or $b \in I$.*

Definition 3.10 *A proper two sided ideal I of a ring R is called a right N weakly primary provided $a, b \in R$ with $0 \neq ab \in I$ implies $b \in I$ or $a \in N(I)$. The ideal I is called a left N weakly primary provided $a, b \in R$ with $0 \neq ab \in I$ implies $a \in I$ or $b \in N(I)$. The ideal I is said to be N weakly primary provided it is both right and left N weakly primary.*

If in definition 3.10 we substitute the symbols P and J for N we have the definitions of *right P weakly primary*, *left P weakly primary*, *P weakly primary*, *right J weakly primary*, *left J weakly primary* and *J weakly primary*.
Note that, if P is a two sided ideal, then $N \subseteq P \subseteq J$ and it follows that:
N weakly primary $\Rightarrow$ P weakly primary $\Rightarrow$ J weakly primary.

Theorem 3.11 *Let R be a ring. Then the following are hold:*
(1) If $x \in R$ is a right unit, then x is a unit.
(2) If xy is a right unit, then both x and y are units.

Proof:
(1) Assume that $xu = 1_R$ for some $u \in R$, then $(1 - ux)u = 0$, and since 0 is always a J weakly primary ideal of R, then $1 - ux \in J$, which implies $ux = 1 + t$ for some $t \in J$. Hence ux is a unit. Thus x is a unit.
(2) Follows immediately from (1).

Theorem 3.12 *A two sided ideal I of a duo ring R is completely weakly prime iff $N \subseteq I$ and I/N is completely weakly prime ideal of R/N.*

Proof:
Let Φ be the natural homomorphism of R onto R/N such that if I is any subset of R, let I' denote the subset $I\Phi$ of R/N. Thus $R' = R/N$. If $x \in N$, we have $x^n = 0 \in I$ for some positive integer n. Hence $x \in I$ which implies that $N \subseteq I$. If now, $N \subseteq I$, then $R'/I' \cong R/I$. So we conclude that I is completely weakly prime ideal in R iff I' is completely weakly prime ideal in R'.

Recall that P is not in general P a two sided ideal, however we introduce the following definition:

Definition 3.13 *A proper two sided ideal I of a ring R is said to be right P weakly primary provided $a, b \in R$ with $0 \neq ab \in I$ implies $b \in I$ or $a^n \in I$ for some positive integer n.*

In order to discuss the right P weakly primary ideals we impose the following three conditions:
(i) $P(I)$ is a two sided ideal where I is any right P weakly primary two sided ideal of R.(This is true if R satisfies the A.C.C. for right ideals or if R is duo, see [4])
(ii) P=N.
(iii) The nontrivial completely weakly prime two sided ideals of R/N are maximal right ideals.

Theorem 3.14 *Let R be a ring satisfying the three conditions (i),(ii) and (iii) above. Then the non trivial not nil two sided ideal I is a completely weakly prime ideal iff I is a maximal right ideal.*

Proof:
Let I be a non trivial not nil completely weakly prime two sided ideal of R. If I is completely weakly prime, then from Theorem 3.13, we have I' is completely weakly prime and non trivial. Hence I' is a maximal right ideal of R'. Thus I is a two sided ideal which is maximal right ideal by Theorem 3.13.

Theorem 3.15 *Let R be a duo ring and let I be a two sided P weakly primary ideal that is not P primary over R. Then $I \subseteq J$ where J is the Jacobson radical of R.*

Proof:
Let $x \in I$. We may assume that $x \neq 0$. It is sufficient to show that $yx + 1$ is a unit of R for every $y \in R$. Since I is P weakly primary ideal that is not P primary over R, so $I^2 = 0$. (by Lemma 3.2) $\Rightarrow 1 = 1 - y^2x^2 = (1 + yx)(1 - yx)$ $\Rightarrow 1 + yx$ is a unit of R. (by Theorem 3.11) $\Rightarrow x \in J$.

ACKNOWLEDGEMENTS. The authors thanks the referee for useful comments.

References

[1] D.D. Anderson and E. Smith, Weakly prime ideals, Houston J.Math., **29**, (2003), 831-840.

[2] A. Ashour, Primary finitely compactly packed modules and $S-$Avoidance theorem for modules, Turkish Journal of Mathematics, **32**, (2008), 315-324.

[3] S.E. Atani and F. Frazalipour, On weakly primary ideals, Georgian Mathematical Journal. **12**, (2005), 423-429.

[4] E.H. Feller, Properties of primary noncommutative rings, University of Wisconsin-Milwaukee. **28**,(1958), 79-91.

[5] E.H. Feller, The lattice of submodules of a module over a commutative rings, Trans.Amer.Math.Soc., **81**, (1956), 342-357.

[6] J.S. Golan, Semirings and their applications, Kluwer Academic Publishers. Dordrecht(1999).

[7] V. Gupta and J.N. Chaudhari, Characterization of weakly prime subtractive ideals in semirings, Bulletin of Institute of Mathematics-Academia Sinica., **3**, (2008), 347-352.

[8] Y. Hirano, E. Poon, and H. Tsutsui, On rings in which every ideals is weakly prime, Bull.Korean Math.Soc., **47**, (2010), 1077-1087.

[9] J. Levitski, On multiplicative systems, Compositio Math., **8**, (1950), 76-80.

Received: February, 2012