Variational Iteration Method for Solving Nonlinear Coupled Equations in 2-Dimensional Space in Fluid Mechanics

A. A. Hemeda

Abstract

In this paper, we shall use the variational iteration method to introduce a framework for analytic treatment of linearized Burgers’ equation and coupled Burgers’ equations in 2-dimensional space. For these equations, the exact solutions are obtained. The paper confirms the power and efficiency of the method in reducing the size of the calculation, also the simplicity of the method compared with the other methods.

Keywords: Linearized Burgers’ equation; Coupled Burgers’ equations; Variational iteration method; Lagrange multipliers

1 Introduction

The variational iteration method, proposed by Ji-Huan He [1-13] in 1999, is used widely by many authors to solve linear and nonlinear physical and mathematical problems. The efficiency of the method and the reduction in the size of computational domain gave this method a wider applicability [14-16]. In the past several decades, many more of the methods such as finite difference methods, perturbation methods, and inverse scattering method were used to solve these problems but with high expensive, hard work and low accuracy. Recently, the variational iteration method is used excellently in place of these methods with low expensive, simple work and high accuracy.

The numerical solution of Burgers’ equation is of great importance due to the equation’s application in the approximate theory of flow through a shock wave traveling in a viscous fluid [17] and is obtained by several researchers, for more details see [16]. For simplicity, the linearized Burgers’ equation is often used in place of Burgers’ equation. The coupled Burgers’ equations, derived
by Esipov [18], is a simple model of evolution of scaled volume concentration of two kinds of particles in fluid suspensions or colloids under the effect of gravity [19].

In this paper we shall solve the linearized Burgers’ equation in addition to the coupled homogenous and inhomogenous Burgers’ equations in 2-dimensional space [16,20]. The paper shows the efficiency and the simplicity of the variational iteration method comparable with the other methods.

The paper is organized as: in section 2 we describe the proposed method. In section 3 numerical problems are used to show the efficiency of the method. Finally, in section 4 conclusions are given.

2 The variational iteration method

To illustrate the basic concepts of the variational iteration method, we consider the following differential equation:

$$Lu + Nu = g(x),$$  \hspace{1cm} (1)

where $L$ is a linear operator, $N$ a nonlinear operator and $g(x)$ an inhomogenous term.

According to the variational iteration method, we can construct the correction functional for (1) as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda [L u_n(\xi) + N \tilde{u}_n(\xi)] d\xi,$$  \hspace{1cm} (2)

where $\lambda$ is a general Lagrange multiplier [1-13], which can be identified optimally via the variational theory, the subscript $n$ denotes the $n$th-order approximation and $\tilde{u}_n$ is considered as a restricted variation [1-13], i.e. $\delta \tilde{u}_n = 0$.

To illustrate the powerful of the variational iteration method, three problems of special interest are discussed in details in the following section.

3 Application problems

Problem 3.1. 2- dimensional linearized Burgers’ equation

For the purpose of illustration of the variational iteration method for solving the linearized Burgers’ equation in 2-dimensional space, we will consider the following equation:
Variational iteration method

\[ u_t + c \nabla u = \mu \nabla^2 u, \quad (3) \]

with the initial conditions:

\[ u(0, y, t) = \exp(-2\mu k^2 t) \sin k(y - 2ct), \quad (4a) \]

\[ u_x(0, y, t) = k \exp(-2\mu k^2 t) \cos k(y - 2ct), \quad (4b) \]

where \( c \) is a free parameter. If \( \mu = 0 \), the first-order wave equation in 2-dimension is obtained. If \( c = 0 \), the 2-dimensional heat equation is obtained [20].

The correction functional for (3) in x-direction is:

\[
u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^x \lambda \left[ \frac{\partial \tilde{u}_n(\xi, y, t)}{\partial t} + c \left( \frac{\partial \tilde{u}_n(\xi, y, t)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi, y, t)}{\partial y} \right) - \mu \left( \frac{\partial^2 u_n(\xi, y, t)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi, y, t)}{\partial y^2} \right) \right] d\xi. \quad (5)\]

The stationary conditions for (5) can be obtained as follows:

\[ \mu \lambda''(\xi) = 0, \quad 1 + \mu \lambda'(\xi)|_{\xi=x} = 0, \quad \mu \lambda(\xi)|_{\xi=x} = 0. \quad (6) \]

The Lagrange multiplier, therefore, can be identified as:

\[ \lambda = \frac{x - \xi}{\mu}. \quad (7) \]

Substituting this value into the functional (5) gives the iteration formula:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) + \frac{1}{\mu} \int_0^x (x - \xi) \left[ \frac{\partial u_n(\xi, y, t)}{\partial t} + \right. \]

\[ + \]
\[
c \left( \frac{\partial u_n(\xi, y, t)}{\partial \xi} + \frac{\partial u_n(\xi, y, t)}{\partial y} \right) - \mu \left( \frac{\partial^2 u_n(\xi, y, t)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi, y, t)}{\partial y^2} \right) \right] d\xi.
\]

(8)

Starting with the zeroth approximation: \( u_0 = A + Bx \), where \( A \) and \( B \) are functions in \( y \) and \( t \) to be determined by using the initial conditions (4), thus we can obtain:

\[
u_0(x, y, t) = \exp(-2\mu k^2 t)(\sin k(y - 2ct) + kx \cos k(y - 2ct)).
\]

(9)

Using (9) into (8) we obtain the first approximation:

\[
u_1(x, y, t) = u_0(x, y, t) + \frac{\exp(-2\mu k^2 t)}{\mu} \int_0^x (x - \xi) \left[ c k^2 \sin k(y - 2ct)\xi - \mu k^3 \cos k(y - 2ct)\xi - \mu k^2 \sin k(y - 2ct)\xi \right] d\xi = u_0(x, y, t) + \frac{\exp(-2\mu k^2 t)}{\mu} \int_0^x \left[ c k^2 x \sin k(y - 2ct)\xi - c k^2 \sin k(y - 2ct)\xi^2 - \mu k^3 x \cos k(y - 2ct)\xi + \mu k^3 \cos k(y - 2ct)\xi^2 - \mu k^2 x \sin k(y - 2ct)\xi + \mu k^2 \sin k(y - 2ct)\xi \right] d\xi =
\exp(-2\mu k^2 t) \left[ \left( 1 - \frac{k^2 x^2}{2} + \frac{c k^2 x^3}{6\mu} \right) \sin k(y - 2ct) + \left( kx - \frac{k^3 x^3}{6} \right) \cos k(y - 2ct) \right].
\]

(10)

In the same manner, we can obtain using the Maple Package, the following successive approximations:

\[
u_0(x, y, t) = \exp(-2\mu k^2 t)(\sin k(y - 2ct) + kx \cos k(y - 2ct)),
\]
\[ u_1(x, y, t) = \exp(-2\mu k^2 t) \left[ \left(1 - \frac{(kx)^2}{2!} + \frac{ck^2 x^3}{6\mu}\right) \sin k(y - 2ct) + \left(kx - \frac{(kx)^3}{3!}\right) \cos k(y - 2ct) \right], \]

\[ u_2(x, y, t) = \exp(-2\mu k^2 t) \left[ \left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{ck^4 x^5}{60\mu} + \frac{c^2 k^2 x^4}{24\mu^2}\right) \sin k(y - 2ct) + \left(kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{c^2 k^3 x^5}{120\mu^2}\right) \cos k(y - 2ct) \right], \]

\[ u_n(x, y, t) = \exp(-2\mu k^2 t) \left[ \left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + \ldots\right) \sin k(y - 2ct) + \left(kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + \ldots\right) \cos k(y - 2ct) \right]. \] (11)

In closed form the solution \( u(x, y, t) \) takes the form:

\[ u(x, y, t) = \exp(-2\mu k^2 t) \sin k(x + y - 2ct), \] (12)

which satisfies exactly the linearized Burgers’ equation (3) and the initial conditions (4).

**Problem 3.2. 2- dimensional homogenous coupled Burgers’ equations**

For the purpose of illustration of the variational iteration method for solving the homogenous form of coupled Burgers’ equations in 2- dimensional space, we will consider the following system of equations [16,20]:

\[ u_t - \nabla^2 u - 2u \nabla u + (uv)_x + (uv)_y = 0, \] (13)
\[ v_t - \nabla^2 v - 2v \nabla v + (uv)_x + (uv)_y = 0, \quad (14) \]

with the initial conditions:

\[ u(x, y, 0) = \cos(x + y), \quad v(x, y, 0) = \cos(x + y). \quad (15) \]

The correction functional for equations (13) and (14) in t- direction are:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda_1 \left[ (u_n)_{\tau} - \nabla^2 \tilde{u}_n - 2\tilde{u}_n \nabla \tilde{u}_n + (\tilde{u}_n v_n)_{x} + (\tilde{u}_n v_n)_{y} \right] d\tau, \quad (16) \]

\[ v_{n+1}(x, y, t) = v_n(x, y, t) + \int_0^t \lambda_2 \left[ (v_n)_{\tau} - \nabla^2 v_n - 2v_n \nabla v_n + (\tilde{u}_n v_n)_{x} + (\tilde{u}_n v_n)_{y} \right] d\tau. \quad (17) \]

The stationary conditions for equations (16) and (17) can be obtained as follows:

\[ \lambda'_1(\tau) = 0, \quad 1 + \lambda_1(\tau)|_{\tau=t} = 0; \quad \lambda'_2(\tau) = 0, \quad 1 + \lambda_2(\tau)|_{\tau=t} = 0. \quad (18) \]

The Lagrange multipliers, therefore, can be identified as:

\[ \lambda_1 = \lambda_2 = -1. \quad (19) \]

Substituting (19) into the functionals (16) and (17), give the iteration formulae:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t [(u_n)_{\tau} - \nabla^2 u_n - \ldots \ldots] \]
\[ 2u_n \nabla u_n + (u_n v_n)_x + (u_n v_n)_y \] d\tau, \quad (20) \\

\[ v_{n+1}(x, y, t) = v_n(x, y, t) - \int_0^t [(v_n)_x - \nabla^2 v_n - \\
\] \\
\[ 2v_n \nabla v_n + (u_n v_n)_x + (u_n v_n)_y] d\tau. \quad (21) \]

Starting with the zeroth approximations: 

\[ u_0(x, y, t) = u(x, y, 0) = \cos(x + y) \] and 

\[ v_0(x, y, t) = v(x, y, 0) = \cos(x + y), \] we can obtain the following first approximations:

\[ u_1(x, y, t) = u_0(x, y, t) - \int_0^t [0 + 2 \cos(x + y) + 4 \cos(x + y) \sin(x + y) - \\
\] \\
\[ 4 \cos(x + y) \sin(x + y)] d\tau = u_0(x, y, t) - 2 \cos(x + y) \int_0^t d\tau = \\
\] \\
\[ \cos(x + y) (1 - 2t), \quad (22) \]

\[ v_1(x, y, t) = v_0(x, y, t) - \int_0^t [0 + 2 \cos(x + y) + 4 \cos(x + y) \sin(x + y) - \\
\] \\
\[ 4 \cos(x + y) \sin(x + y)] d\tau = u_0(x, y, t) - 2 \cos(x + y) \int_0^t d\tau = \\
\] \\
\[ \cos(x + y) (1 - 2t). \quad (23) \]
In the same manner, we can obtain using the Maple Package, the following successive approximations:

\[
\begin{align*}
  u_0(x, y, t) &= \cos(x + y), \\
  v_0(x, y, t) &= \cos(x + y), \\
  u_1(x, y, t) &= \cos(x + y)(1 - 2t), \\
  v_1(x, y, t) &= \cos(x + y)(1 - 2t), \\
  u_2(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!}\right), \\
  v_2(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!}\right), \\
  u_3(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!}\right), \\
  v_3(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!}\right), \\
  \vdots \\
  u_n(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \ldots \right), \\
  v_n(x, y, t) &= \cos(x + y) \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \ldots \right).
\end{align*}
\]

In closed form the solutions \(u(x, y, t)\) and \(v(x, y, t)\) take the forms:

\[
\begin{align*}
  u(x, y, t) &= \exp(-2t) \cos(x + y), \\
  v(x, y, t) &= \exp(-2t) \cos(x + y).
\end{align*}
\]
which are the exact solutions satisfies the homogenous coupled Burgers’ equations (13) and (14) and the initial conditions (15).

Problem 3.3. 2- dimensional inhomogenous coupled Burgers’ equations

For the purpose of illustration of the variational iteration method for solving the inhomogenous form of coupled Burgers’s equations in 2- dimensional space, we will consider the following system of equations [16,20]:

\[
\begin{align*}
\frac{du}{dt} - \nabla^2 u - 2u \nabla u + (uv)_x + (uv)_y &= f(x, y, t), \\
\frac{dv}{dt} - \nabla^2 v - 2v \nabla v + (uv)_x + (uv)_y &= g(x, y, t),
\end{align*}
\]

with the initial conditions:

\[
\begin{align*}
u(x, y, 0) &= \exp(-x - y), & v(x, y, 0) &= \exp(-x - y). \tag{30}
\end{align*}
\]

The correction functional for equations (28) and (29) in t- direction are:

\[
\begin{align*}
&u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda_1 \left[ (u_n)_\tau - \nabla^2 \bar{u}_n - 2\bar{u}_n \nabla \bar{u}_n + \\
&(\bar{u}_n \bar{v}_n^\sim)_x + (\bar{u}_n \bar{v}_n^\sim)_y - f(x, y, \tau) \right] d\tau = 0, \tag{31}
\end{align*}
\]

\[
\begin{align*}
v_{n+1}(x, y, t) = v_n(x, y, t) + \int_0^t \lambda_2 \left[ (v_n)_\tau - \nabla^2 \bar{v}_n - 2\bar{v}_n \nabla \bar{v}_n + \\
&(\bar{u}_n \bar{v}_n^\sim)_x + (\bar{u}_n \bar{v}_n^\sim)_y - g(x, y, \tau) \right] d\tau = 0. \tag{32}
\end{align*}
\]

Following the same procedure as problem 3.2., the Lagrange multipliers are:

\[
\lambda_1 = \lambda_2 = -1. \tag{33}
\]

Substituting these values into the functionals (31) and (32) give the iteration formulae:
\[
    u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t [(u_n)_\tau - \nabla^2 u_n - 2u_n \nabla u_n + \\
    (u_nv_n)_x + (u_nv_n)_y - f(x, y, \tau)] \, d\tau = 0, \tag{34}
\]

\[
    v_{n+1}(x, y, t) = v_n(x, y, t) - \int_0^t [(v_n)_\tau - \nabla^2 v_n - 2v_n \nabla v_n + \\
    (u_nv_n)_x + (u_nv_n)_y - g(x, y, \tau)] \, d\tau = 0. \tag{35}
\]

Starting with the zeroth approximations: \( u_0(x, y, t) = u(x, y, 0) = \exp(-x-y) \) and \( v_0(x, y, t) = v(x, y, 0) = \exp(-x-y) \), we can obtain the following first approximations:

\[
    u_1(x, y, t) = u_0(x, y, t) - \int_0^t [0 - 2e^{-x-y} + 4e^{-2x-2y} - 4e^{-2x-2y} + \\
    e^{-x-y} \sin \tau + 2e^{-x-y} \cos \tau] \, d\tau = u_0(x, y, t) - \exp(-x-y).\]

\[
    \int_0^t [\sin \tau + \cos \tau - 2] \, d\tau = \exp(-x-y) [\cos t - \sin t + 2t], \tag{36}
\]

\[
    v_1(x, y, t) = v_0(x, y, t) - \int_0^t [0 - 2e^{-x-y} + 4e^{-2x-2y} - 4e^{-2x-2y} + \\
    e^{-x-y} \sin \tau + 2e^{-x-y} \cos \tau] \, d\tau = v_0(x, y, t) - \exp(-x-y).\]
\[
\int_0^t [\sin \tau + \cos \tau - 2] \, d\tau = \exp(-x - y) [\cos t - \sin t + 2t].
\]

(37)

In the same manner, we can obtain using the Maple Package, the following successive approximations:

\[
\begin{align*}
  u_0(x, y, t) &= \exp(-x - y), \\
  v_0(x, y, t) &= \exp(-x - y), \\
  u_1(x, y, t) &= \exp(-x - y) [\cos t + 2(t - \sin t)], \\
  v_1(x, y, t) &= \exp(-x - y) [\cos t + 2(t - \sin t)], \\
  u_2(x, y, t) &= \exp(-x - y) \left[ 5 \cos t - 4 \left( 1 - \frac{t^2}{2!} \right) \right], \\
  v_2(x, y, t) &= \exp(-x - y) \left[ 5 \cos t - 4 \left( 1 - \frac{t^2}{2!} \right) \right], \\
  u_3(x, y, t) &= \exp(-x - y) \left[ \cos t + 8 \left( \sin t - \left( t - \frac{t^3}{3!} \right) \right) \right], \\
  v_3(x, y, t) &= \exp(-x - y) \left[ \cos t + 8 \left( \sin t - \left( t - \frac{t^3}{3!} \right) \right) \right], \\
  u_4(x, y, t) &= \exp(-x - y) \left[ 16 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \right) - 15 \cos t \right], \\
  v_4(x, y, t) &= \exp(-x - y) \left[ 16 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \right) - 15 \cos t \right], \\
  u_5(x, y, t) &= \exp(-x - y) \left[ \cos t + 32 \left( \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} \right) - \sin t \right) \right], \\
  v_5(x, y, t) &= \exp(-x - y) \left[ \cos t + 32 \left( \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} \right) - \sin t \right) \right],
\end{align*}
\]
\[ u_6(x, y, t) = \exp(-x - y) \left[ 65 \cos t - 64 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right) \right], \]
\[ v_6(x, y, t) = \exp(-x - y) \left[ 65 \cos t - 64 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right) \right], \]
\[ u_7(x, y, t) = \exp(-x - y) \left[ 65 \cos t - 64 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right) \right], \]
\[ v_7(x, y, t) = \exp(-x - y) \left[ 65 \cos t - 64 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right) \right], \]
\[ u_8(x, y, t) = \exp(-x - y) \left[ 256 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \right) - 255 \cos t \right], \]
\[ v_8(x, y, t) = \exp(-x - y) \left[ 256 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \right) - 255 \cos t \right], \]
\[ u_{4n+1}(x, y, t) = \exp(-x - y) \left[ \cos t + 2^{n+1} \left( \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \right) - \sin t \right) \right], \]
(38a)
\[ u_{4n+2}(x, y, t) = \exp(-x - y) \left[ (2^{n+2} + 1) \cos t - 2^{n+2} \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \right) \right], \]
(38b)
\[ u_{4n+3}(x, y, t) = \exp(-x - y) \left[ \cos t + 2^{n+3} \left( \sin t - \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \right) \right) \right], \]
(38c)
\[ u_{4n+4}(x, y, t) = \exp(-x - y) \left[ 2^{n+4} \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \right) - (2^{n+4} - 1) \cos t \right], \]
(38d)
\[ v_{4n+1}(x, y, t) = \exp(-x - y) \left[ \cos t + 2^{n+1} \left( \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \right) - \sin t \right) \right], \]
(39a)
\[ v_{4n+2}(x, y, y) = \exp(-x - y) \left[ (2^{n+2} + 1) \cos t - 2^{n+2} \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \right) \right], \quad (39b) \]

\[ v_{4n+3}(x, y, t) = \exp(-x - y) \left[ \cos t + 2^{n+3} \left( \sin t - \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \right) \right) \right], \quad (39c) \]

\[ v_{4n+4}(x, y, t) = \exp(-x - y) \left[ 2^{n+4} \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \right) - (2^{n+4} - 1) \cos t \right], \quad (39d) \]

where in both (38) and (39) \( n = 0, 1, 2, 3, \ldots \). When \( n = 0 \), we obtain the successive approximations \( u_1, u_2, u_3, u_4, v_1, v_2, v_3 \) and \( v_4 \) and so on.

In closed form the solutions \( u(x, y, t) \) and \( v(x, y, t) \) take the forms:

\[ u(x, y, t) = \exp(-x - y) \cos t, \quad (40) \]

\[ v(x, y, t) = \exp(-x - y) \cos t, \quad (41) \]

which are the exact solutions satisfies the inhomogenous coupled Burgers’ equations (28) and (29) and the initial conditions (30).

4 Conclusion

In this paper, the variational iteration method has been successfully used to finding the solution of a linearized Burgers’ equation, homogenous coupled Burgers’ equations and inhomogenous coupled Burgers’ equations in 2-dimensional space. The solution obtained by the mentioned method is an infinite series for appropriate initial conditions, which can in turn, be expressed in a closed form, the exact solution. The obtained results show that the variational iteration method is a powerful method to solving linearized Burgers’ and coupled Burgers’ equations in 2-dimensional space, it is also a promising method to solve other equations.

References

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