Multiplication Operators on Weighted Spaces of Continuous Functions with Operator-Valued Weights

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Abstract

In this paper, we characterize bounded and invertible multiplication operators on the weighted spaces of continuous functions with operator-valued weights.

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1 Introduction:

Let $X$ be a non-empty set and $H$ be a Hilbert space. Let $L(X, H) = \{ f : f : X \rightarrow H \}$ be the topological vector space of $H$-valued functions defined on $X$. If $\phi : X \rightarrow B(H)$ is an operator-valued map, then a continuous linear transformation $M_\phi : L(X, H) \rightarrow L(X, H)$ defined by $M_\phi(f) = \phi(x)(f(x))$ for every $x \in X, f \in L(X, H)$ is called a multiplication operator.

Nachbin [3] initiated a study of the weighted spaces followed by Prolla [1, 2], Summers [7] for a variety of problems such as weighted approximation problem, description of tensor product etc. An intensive study of multiplication
operators on weighted spaces is made by Singh and Manhas [4], [5] and [6].
In the present paper, we characterize bounded and invertible Multiplication operators on weighted spaces of continuous functions with operator-valued weights.

2 Preliminaries:

Let $X$ be a completely regular Hausdorff space. Let $B(H)$ denote the Banach algebra of all bounded linear operators from $H$ into itself. By a weight on $X$, we shall mean a strongly upper semi continuous positive operator valued map from $X$ to $B(H)$. A weight $v : X \to B(H)$ is called positive if and only if $\langle v(x)y, y \rangle \geq 0$ for every $x \in X$ and $y \in H$ and we write $v \geq 0$. For $y \in H$, $s_y ov : X \to \mathbb{R}$ is a map defined by $(s_y ov)(x) = ||v(x)y||$ for all $x \in X$.

We say that $v : X \to B(H)$ is strongly upper semi continuous if the set $\{x \in X : ||v(x)y|| < a\} = (s_y 0v)^{-1} (-\infty, a)$ is open for all $a \in \mathbb{R}$. Let $V$ denote a set of weights on $X$. We assume that for every $x \in X$, $\bigcap_{v \in V} Ker v(x) = \{0\}$.

A set $V$ of weights on $X$ is said to be directed upward if for every pair $v_1, v_2 \in V$ and $\lambda > 0$ there exists $v \in V$ such that $\lambda v_1, \lambda v_2 \leq v$. We say that $v f$ vanishes at infinity on $X$ if for each $\epsilon > 0$, the set $\{x \in X : ||v(x)f(x)|| \geq \epsilon\}$ is compact for all $v \in V$.

We shall now introduce the weighted spaces of continuous functions on $X$ with respect to system of operator valued weights $V$.

$CV_0(X, H) = \{f \in C(X, H) : v f \text{ vanishes at infinity on } X \text{ for all } v \in V\}.
CV_b(X, H) = \{f \in C(X, H) : v f(X) \text{ is bounded in } H \text{ for all } v \in V\}.
CV_p(X, H) = \{f \in C(X, H) : v f(X) \text{ is pre compact in } H \text{ for all } v \in V\}.$

Clearly $CV_0(X, H), CV_b(X, H)$ and $CV_p(X, H)$ are vector spaces. Now for all $v \in V$ and $f \in C(X, H)$, we put

$$p_v(f) = sup\{||v(x)(f(x))|| : x \in X\}.$$

Then $p_v$ is a semi norm. The family $\{p_v : v \in V\}$ of semi norms defines a locally convex Hausdorff topology on each of these spaces. We denote this topology by $W_v$ and the above vector spaces endowed with $W_v$ are called weighted spaces of continuous functions with operator-valued weights. It has a basis of closed absolutely convex neighbourhoods of the origin of the form

$$B_v = \{f \in CV_b(X, H) : ||f||_v \leq 1\}.$$

If $U$ and $V$ are the two systems of weights on $X$, then we write $U \leq V$, whenever given $u \in U$ there exists $v \in V$ such that $u(x) \leq v(x)$ for every $x \in X$. 
3 A Characterization of Multiplication Operator on $CV_b(X, H)$:

In this section we present a characterization of multiplication operator. The symbol $(V \phi)^*(V \phi) \leq V^*V$ denotes the family $\{(v \phi)^*(v \phi) \leq v^*v : v \in V\}$.

**Theorem 3.1** Let $\phi : X \rightarrow B(H)$ be continuous. Then $M_\phi : CV_b(X, H) \rightarrow CV_b(X, H)$ is continuous if and only if $(V \phi)^*(V \phi) \leq V^*V$.

**Proof**: We first assume that $(V \phi)^*(V \phi) \leq V^*V$. That is, for every $v \in V$ there exists $M$ such that $(v \phi)^*(v \phi) \leq u^*u$ or equivalently

$$ ||v(x)(\phi(x)y)|| \leq ||u(x)y|| \quad \forall \ x \in X, y \in H. $$

Let $\{x_\alpha\}$ be a net in $X$ converging to $x \in X$. Then for $f \in CV_b(X, H)$,

$$ ||(M_\phi f)(x_\alpha) - (M_\phi f)(x)|| \leq ||(\phi(x_\alpha) - \phi(x))f(x_\alpha)|| + ||\phi(x)(f(x_\alpha) - f(x))|| \leq ||\phi(x_\alpha) - \phi(x)||M + ||\phi(x)||||f(x_\alpha) - f(x)|| \rightarrow 0 $$

where $M$ is such that $||f(x_\alpha)|| \leq M$ for every $\alpha$. Hence $M_\phi f$ is continuous and $M_\phi f \in CV_b(X, H)$. Further it is sufficient to show that $M_\phi$ is continuous at the origin. For this, suppose $\{f_\alpha\}$ be a net in $CV_b(X, H)$ such that $||f_\alpha||_v \rightarrow 0$ for every $v \in V$. Now

$$ ||M_\phi f_\alpha||_v = \sup\{||v(x)(\phi(x)f_\alpha(x))|| : x \in X\} \leq \sup\{||u(x)f_\alpha(x)|| : x \in X\} = ||f_\alpha||_u \rightarrow 0 $$

This proves the continuity of $M_\phi$ at the origin.

Conversely, suppose $M_\phi$ is a continuous linear operator on $CV_b(X, H)$. Let $v \in V$, since $M_\phi$ is continuous at origin, so there exists $u \in V$ such that $M_\phi(B_u) \subseteq B_v$. We claim that

$$ ||v(x)(\phi(x)y)|| \leq 2||u(x)y|| \quad \forall \ x \in X, y \in H. $$

Take $x_\alpha \in X$ and $y_\alpha \in H$. Set $||u(x_\alpha)y_\alpha|| = \epsilon$. In case $\epsilon > 0$ ,

the set $G = \{x \in X : ||u(x)y_\alpha|| < 2\epsilon\}$ is an open neighbourhood of $x_\alpha$. By Nachbin lemma [ 3, Lemma 2, p. 69] there exists $f \in C||V||_b(X, R)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$, where $C||V||_b(X, R) = \{f : f : X \rightarrow R$ is continuous and $\sup\{||v(x)||f(x)|| : x \in X\} < \infty \ \forall \ v \in V\}$. Define $g : X \rightarrow H$ by $g(x) = f(x)\phi_\alpha$ for all $x \in X$. Then $g \in CV_b(X, H)$ and $||g(x_\alpha)|| = ||f(x)y_\alpha|| = ||y_\alpha||$ and $g(X - G) = 0$. Let $h = \frac{g}{2||u(x_\alpha)y_\alpha||}$. Then

$$ ||u(x)h(x)|| \leq \frac{||u(x)y_\alpha||}{2||u(x_\alpha)y_\alpha||} \leq \frac{2\epsilon}{2\epsilon} = 1. $$
or that \( h \in B_u \) and hence \( \phi h \in B_v \) because \( M_\phi(B_u) \subseteq B_v \).

Thus \( ||v(x)(\phi(x)h(x))|| \leq 1 \), for every \( x \in X \). Hence \( ||v(x_o)(\phi(x_o)y_o)|| \leq 2||u(x_o)y_o|| \).

If \( ||u(x_o)y_o|| = 0 \), then we show that \( ||v(x_o)(\phi(x_o)y_o)|| = 0 \).

Suppose if possible \( ||v(x_o)(\phi(x_o)y_o)|| > 0 \). Put \( \epsilon = ||v(x_o)(\phi(x_o)y_o)|| \).

Then \( G = \{ x \in X : ||u(x)y_o|| < \epsilon \} \) is an open neighbourhood of \( x_o \). So by Nachbin lemma [3, lemma 2, p. 69] there exists \( f \in C||V||b(X,R) \) such that \( 0 \leq f \leq 1 \), \( f(x_o) = 1 \) and \( f(X - G) = 0 \). Define \( g : X \rightarrow H \) as \( g(x) = f(x)y_o \) for all \( x \in X \). Then

\[
g \in CV_o(X,H).
\]

Let \( h = \frac{1}{\epsilon} g \). Then clearly \( h \in B_u \) and therefore \( \phi h \in B_v \). This implies that

\[
||v(x)(\phi(x)h(x))|| \leq 1, \text{ for every } x \in X.
\]

From this, it follows that

\[
||v(x_o)(\phi(x_o)y_o)|| \leq \epsilon = \frac{||v(x_o)(\phi(x_o)y_o)||}{2}.
\]

Which is impossible. So \( ||v(x_o)(\phi(x_o)y_o)|| = 0 \). This proves our inequality.

\section{Invertible Multiplication Operators}

A characterization of the invertible multiplication operator is reported in this section.

\textbf{Theorem 4.1} Let \( H \) be a Hilbert space and let \( \phi : X \rightarrow B(H) \) be a continuous mapping such that \( M_\phi \in B(CV_o(X,H)) \). Then \( M_\phi \) has dense range if and only if the operator \( M_{\phi(x)} \) has dense range in \( H \) for every \( x \in X \), where \( M_{\phi(x)} : H \rightarrow H \) is defined as \( M_{\phi(x)}(y) = \phi(x)y \ \forall \ x \in X, y \in H \).

\textbf{Proof} : Suppose first that \( M_\phi : CV_o(X,H) \rightarrow CV_o(X,H) \) has dense range.

We shall prove that for every \( x \in X, M_{\phi(x)} : H \rightarrow H \) has dense range. Let \( z \in H \) and \( V_z \) be an open neighbourhood of \( z \). We can find \( f \in C||V||b(X,R) \) such that \( 0 \leq f \leq 1 \), \( f(z) = 1 \) and \( f(X - V_z) = 0 \), where \( C||V||b(X,R) = \{ f : X \rightarrow R \text{ is continuous and for each } \epsilon > 0, \text{ there exists a compact subset } K \text{ of } X \text{ such that } ||v(x)|| ||f(x)|| < \epsilon \ \forall \ x \in X - K \} \).

Define \( g_z : X \rightarrow H \) by \( g_z(x) = f(x)z \ \forall \ x \in X \). Then \( g_z \in CV_o(X,H) \). Now

\[
g_z \in \{ M_{\phi}h : h \in CV_o(X,H) \}
\]

if and only if \( g_z(z) = z \in \{ \phi(z)h(z) : h \in CV_o(X,H) \} \).

if and only if \( \text{ran } \phi(z) \text{ is dense in } H \).

Conversely, suppose \( M_{\phi(x)} : H \rightarrow H \) has dense range for all \( x \in X \). Let \( f \in CV_o(X,H) \) and \( x_0 \in X \). Since \( M_{\phi(x)} \) has dense range, we can find a sequence \( \{ y_n \} \subseteq H \) such that \( \phi(x_0)y_n \rightarrow f(x_0) \). Let \( U \) be an open neighbourhood of \( x_0 \),
we can find \( g \in C||V||_0(X, R) \) such that \( 0 \leq \|g\| \leq 1, g(x_0) = 1 \) and \( g(X-U) = 0. \)
Define \( g_n : X \to H \) as \( g_n(z) = g(z) y_n \) for \( z \in X \). Then \( g_n \in CV_0(X, H) \). Now
\[
\lim_n (M_\phi g_n)(x_0) = \lim_n \phi(x_0) g_n(x_0) = \lim_n \phi(x_0) y_n = f(x_0).
\]
Since \( x_0 \) is arbitrary, it follows that \( M_\phi \) has dense range.

**Theorem 4.2** Let \( \phi : X \to B(H) \) be a continuous mapping. Then \( M_\phi : CV_0(X, H) \to CV_0(X, H) \) is bounded from below if and only if \( V^* V \leq (V^* V) o \phi \).

**Proof:** We first assume that condition is true. Then
\[
\|v(x)y\| \leq \|u(x)(\phi(x)y)\| \quad \forall \ x \in X, y \in H.
\]
Let \( f \in CV_0(X, H) \). Then
\[
p_v(f) = \sup \{\|v(x)f(x)\| : x \in X\} \\
\leq \sup \{\|u(x)(\phi(x)y)\| : x \in X\} \\
= p_u(M_\phi f).
\]
This implies that if \( f \in B_v^c \) then \( M_\phi f \in B_v^c \). Hence \( M_\phi B_v^c \subseteq B_v^c \). This proves that \( M_\phi \) is bounded from below.

Conversely, suppose that \( M_\phi \) is bounded from below. Then for every \( v \in V \), there exists \( u \in V \) such that \( p_v(f) \leq p_u(M_\phi f) \quad \forall \ f \in CV_0(X, H) \). We claim that
\[
\|v(x)y\| \leq 2\|u(x)(\phi(x)y)\| \quad \forall \ x \in X, y \in H.
\]
Suppose this is false. Then
\[
\|v(x_0)y_0\| > 2\|u(x_0)(\phi(x_0)y_0)\| \quad \text{for some} \quad x_0 \in X, y_0 \in H.
\]
Let \( G = \{x \in X : 2\|u(x)(\phi(x)y_0)\| < \|v(x_0)y_0\|\} \).
Then \( G \) is an open neighbourhood of \( x_0 \). By Nachbin Lemma \( [3, \text{Lemma 2, P. 69}] \), there exists \( f \in C||V||_0(X, R) \) such that \( 0 \leq f \leq 1, f(x_0) = 1 \) and \( f(X - G) = 0. \) Define \( g : X \to H \) by
\[
g(x) = \frac{2f(x)y_0}{\|v(x)y_0\|} \quad \forall \ x \in X.
\]
Then \( g \in CV_0(X, H) \). Also \( p_u(M_\phi f) \leq 1. \) It implies that \( p_v(f) \leq 1. \) In particular, we have, \( \|v(x_0)g(x_0)\| \leq 1. \) But \( \|v(x_0)g(x_0)\| = 2. \) This contradiction proves our claim. This completes the proof of theorem.
Theorem 4.3  Let $V$ be a system of weights on $X$ such that $CV_0(X, H)$ is complete and let $\phi : X \to B(H)$ be a continuous mapping such that $M_\phi \in B(CV_0(X, H))$. Then $M_\phi$ is invertible if and only if

(i) For every $x \in X$, the linear operator $M_{\phi(x)}$ has dense range in $H$.

(ii) $V^*V \leq (V^*V)\circ \phi$.

Proof : The proof follows from theorem 4.1 and theorem 4.2

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References


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