**-Connectedness in Fuzzy Ideal Topological Spaces

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**Abstract.** In the present paper the concept of **-connectedness due to Ekici and Noiri [3] have been extended in fuzzy ideal topological spaces and some of the properties and characterizations of fuzzy **-connected fuzzy ideal topological spaces have been studied.

**Keywords:** Fuzzy ideal topological spaces, fuzzy
**-separated sets and fuzzy **-connectedness

1. Introduction

The fundamental concept of a fuzzy set was introduced by Zadeh [17]. Subsequently, Chang [1] defined the notion of fuzzy topology. Since then various aspects of general topology were investigated and carried out in fuzzy sense by several authors of this field. The local properties of a space which may also be in certain cases the properties of the whole spaces are important field for study in both general and fuzzy topology. In general topology, by introducing the notion of ideal, Kuratowski [8], Vaidyanathaswamy [15,16] and several other authors carried out such analysis. The notion of fuzzy points and Q-neighborhood of a fuzzy point introduces the scope of such analysis in fuzzy topology [11,12].
In 1997 Sarkar [14] extended the concept of ideals in fuzzy settings and defined the local function in fuzzy topology. Since then various fuzzy topological concepts have been studied via fuzzy ideals. In this paper we extend the concept $\ast$-connectedness due to Ekici and Noiri [3] in fuzzy settings and study it in fuzzy ideal topological spaces.

2. Preliminaries

Let $X$ be a nonempty set. A family $\tau$ of fuzzy sets of $X$ is called a fuzzy topology [1] on $X$ if the null fuzzy set $0$ and the whole fuzzy set $1$ belongs to $\tau$ and $\tau$ is closed with respect to any union and finite intersection. If $\tau$ is a fuzzy topology on $X$, then the pair $(X, \tau)$ is called a fuzzy topological space and the members of $\tau$ are called fuzzy open sets and their complements are called fuzzy closed sets. The closure [1] of a fuzzy set $A$ of $X$ denoted by $\text{Cl}(A)$, is the intersection of all fuzzy closed sets which contains $A$. The interior [1] of a fuzzy set $A$ of $X$ denoted by $\text{Int}(A)$ is the union of all fuzzy sets of $X$ contained in $A$. A mapping $f$ from a topological space $(X, \tau)$ to another topological space $(Y, \sigma)$ is said to be fuzzy continuous [1] if the inverse image of every fuzzy open set of $Y$ is fuzzy open in $X$. A fuzzy set $A$ in fuzzy topological space $(X, \tau)$ is said to be quasi-coincident [11] with a fuzzy set $B$ if there exists a point $x \in X$ such that $A(x) + B(x) > 1$. The negation of this statement is written as $\overline{\text{AqB}}$. A fuzzy set $V$ in a fuzzy topological space $(X, \tau)$ is called a Q-neighborhood [11] of a fuzzy point $x_\beta$ if there exists a fuzzy open set $U$ of $X$ such that $x_\beta \text{qU} \subseteq V$.

Lemma 2.1: Let $A$ and $B$ be two fuzzy sets of $X$. Then $A \leq B \iff \overline{\text{Aq}(1-B)}$[11].

A nonempty collection of fuzzy sets $I$ of a set $X$ satisfying the conditions (i) if $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity), (ii) if $A \in I$ and $B \in I$ then $A \vee B \in I$ (finite additivity) is called a fuzzy ideal [14] on $X$. The triplex $(X, \tau, I)$ denotes a fuzzy ideal topological space with a fuzzy ideal $I$ and fuzzy topology $\tau$. The local function [14] for a fuzzy set $A$ of $X$ with respect to $\tau$ and $I$ denoted by $A^*(\tau, I)$ (briefly $A^*$) in a fuzzy ideal topological space $(X, \tau, I)$ is the union of all fuzzy points $X_\beta$ such that if $U$ is a
Q-neighborhood of $x_\beta$ and $E \in I$ then for at least one point $y \in X$ for which $U(y) + A(y) - 1 > E(y)$ [14]. The $*$-closure [14] operator of a fuzzy set $A$ denoted by $\text{Cl}^*(A)$ in $(X, \tau, I)$ defined as $\text{Cl}^*(A) = A \vee A^*$.

In a fuzzy ideal topological space $(X, \tau, I)$, the collection $\tau*(I)$ means an extension of fuzzy topological space finer than $\tau$ via fuzzy ideal which is constructed by considering the Class $\beta = \{U \in I : U \in \tau, E \in I\}$ as a base.

3. Fuzzy $*$-Separated Sets

**Definition 3.1:** Two non empty fuzzy sets $A$ and $B$ of a fuzzy ideal topological space $(X, \tau, I)$ are said to be fuzzy $*$-separated if $\text{Cl}^*(A) \nsubseteq B$ and $A \nsubseteq \text{Cl}^*(B)$.

**Theorem 3.1:** Let $A$ and $B$ be fuzzy $*$-separated sets in a fuzzy ideal topological space $(X, \tau, I)$. If $A_1$ and $B_1$ are two non empty fuzzy sets such that $A_1 \leq A$ and $B_1 \leq B$, then $A_1$ and $B_1$ are fuzzy $*$-separated sets in $X$.

**Proof:** Since $A_1 \leq A$ and $B_1 \leq B$, we have $\text{Cl}^*(A_1) \leq \text{Cl}^*(A)$ and $\text{Cl}(B_1) \leq \text{Cl}(B)$. Therefore $\text{Cl}^*(A) \nsubseteq B$ and $A \nsubseteq \text{Cl}^*(B)$.

**Theorem 3.2:** Let $A$ be a fuzzy open and $B$ is a fuzzy $*$-open in a fuzzy ideal topological space $(X, \tau, I)$. Then $A$ and $B$ are fuzzy $*$-separated in $X$ if and only if $A \nsubseteq B$.

**Proof:** Necessity. If $A \nsubseteq B$ then there exists a point $x \in X$ such that $A(x) + B(x) > 1$. This implies that $\text{Cl}^*(A(x)) + B(x) > 1$ and $A(x) + \text{Cl}(B(x)) > 1$. Hence $\text{Cl}^*(A) \nsubseteq B$ and $A \nsubseteq \text{Cl}(B)$, which is a contradiction. Hence $A \nsubseteq B$.

**Sufficiency.** Let $A \nsubseteq B$, then by Lemma 2.1. $A \leq 1-B$. Therefore $\text{Cl}^*(A) \leq \text{Cl}^*(1-B) = 1-B$ because $1-B$ is fuzzy $*$-closed set in $X$. Hence by Lemma 2.1 $\text{Cl}^*(A) \nsubseteq B$. Similarly $\text{Cl}(B) \nsubseteq A$. Similarly $\text{Cl}^*(A) \nsubseteq B$.

**Theorem 3.3:** Let $A$ be a fuzzy $*$-closed and $B$ is a fuzzy closed set in a fuzzy ideal topological space $(X, \tau, I)$. Then $A$ and $B$ are fuzzy $*$-separated in $X$ if and only if $A \nsubseteq B$.

**Proof:** The proof of this theorem follows from the Definition 3.1 and Lemma 2.1.
Theorem 3.4: Let A be a fuzzy open and B is a fuzzy *-open set in a fuzzy ideal topological space \((X, \tau, I)\). Then the fuzzy sets \(C_A = A \land (1 - B)\) and \(C_B = B \land (1 - A)\) are fuzzy *-separated in \(X\).

Proof: Since \(C_A = A \land (1 - B)\), \(\text{Cl}^*(C_A) \leq \text{Cl}^*(1 - B) = 1 - B\) because B is fuzzy *-open in \(X\). And so by Lemma 2.1, \(\text{Cl}^*(C_A)qB\). Thus \(\text{Cl}^*(C_A)q(C_B)\). Similarly \(\text{Cl}(C_B)q(C_A)\). Hence \(C_A \land C_B\) and \(C_B \land C_A\) are fuzzy *-separated sets in \(X\).

Theorem 3.5: Let A be a fuzzy *-closed and B is a fuzzy closed set in a fuzzy ideal topological space \((X, \tau, I)\). Then the fuzzy sets \(C_A = A \land (1 - B)\) and \(C_B = B \land (1 - A)\) are fuzzy *-separated in \(X\).

Proof: Let Since A is fuzzy *-closed and B is a fuzzy closed in \(X\) , A = \(\text{Cl}^*(A)\) and B = \(\text{Cl}(B)\). Now \(C_A(B) \leq (1 - B)\) we have \(\text{Cl}^*(C_A)qC_B\) and hence \(\text{Cl}(C_A)q(C_B)\). Similarly \(\text{Cl}(C_B)q(C_A)\). Hence \(C_A \land C_B\) and \(C_B \land C_A\) are fuzzy *-separated sets in \(X\).

Theorem 3.6: Let \((X, \tau, I)\) be a fuzzy ideal topological space. Let A and B be two fuzzy *-separated in \(X\) if and only if there exists a fuzzy open set U and a fuzzy *-open V such that \(A \leq U\), \(B \leq V\), \((AqV)\) and \((BqU)\).

Proof: Necessity. Let A and B be fuzzy *-separated fuzzy sets in \(X\). Now put \(V = 1 - \text{Cl}^*(A)\) and \(U = 1 - \text{Cl}(B)\). Then U is fuzzy open and V is fuzzy *-open set in \(X\) such that \(A \leq U\), \(B \leq V\), \((AqV)\) and \((BqU)\).

Sufficiency. Let \(U\) be a fuzzy open and \(V\) be a fuzzy *-open set in \(X\) such that \(A \leq U\), \(B \leq V\), \((AqV)\) and \((BqU)\). Now \(1 - U\) is fuzzy closed and \(1 - V\) is fuzzy *-closed set in \(X\), \(\text{Cl}^*(A) \leq (1 - V) \leq (1 - B)\) and \(\text{Cl}(B) \leq (1 - U) \leq (1 - A)\). Therefore by Lemma 2.1, \((\text{Cl}^*(A)qB)\) and \((\text{Cl}(B)qA)\). Hence A and B are fuzzy *-separated fuzzy sets in \(X\).

4. Fuzzy *-Connectedness

Definition 4.1: A fuzzy set E in a fuzzy ideal topological space \((X, \tau, I)\) is said to be fuzzy *-connected if it cannot be expressed as the union of two fuzzy *-separated sets.
Theorem 4.1: Let A and B be fuzzy *-separated sets in a fuzzy ideal topological space \((X, \tau, I)\) and E be a fuzzy *-connected set in X such that \(E \leq A \lor B\). Then exactly one of the following conditions holds:

(a) \(E \leq A\) and \(E \land B = 0\).
(b) \(E \leq B\) and \(E \land A = 0\).

Proof: We first note that when \(E \land B = 0\) then \(E /\lor A\), since \(E /\lor B\), both \(E \land A = 0\) and \(E \land B = 0\) cannot hold simultaneously. Again if \(E \land B \neq 0\) and \(E \land A \neq 0\). Then \(E \land A\) and \(E \land B\) are fuzzy *-separated sets in X such that \(E = (E \land A) \lor (E \land B)\) contradicting the fuzzy *-connectedness of E. Hence exactly one of the conditions (a) and (b) must holds.

Theorem 4.2: Let \(E, F\) be two fuzzy sets of a fuzzy ideal topological space \((X, \tau, I)\). If \(E\) is fuzzy *-connected and \(E /\lor F\), then \(E\) is fuzzy *-connected.

Proof: If \(E = 0\) then the result is true. Let \(E \neq 0\) and suppose \(F\) is not fuzzy *-connected. Then there exist two fuzzy *-separated sets \(A\) and \(B\) in X such that \(F = A \lor B\). Since \(E\) is fuzzy *-connected and \(E \leq F = E \lor F\), so by Theorem 4.1; \(E \leq A\) and \(E \land B = 0\) or \(E \leq B\) and \(E \land A = 0\). Let \(E \leq A\) and \(E \land B = 0\). Then \(B = B \land F \leq B \land \text{Cl}^*(E) \leq E \land \text{Cl}^*(A) \leq B \land (1-B) \leq B\). It follows that \(B = B \land (1-B)\) and since \(B \neq 0\), \(B(x) = 1/2\) for all \(x \in X\). Thus \(B_0 = X\) where \(B_0\) denotes the support of \(B\). Now \(E \land B = 0\) implies \(E_0 \land B_0 = 0 \Rightarrow E_0 = 0\). Hence, \(E = 0\) which is contradiction. Similarly if \(E \leq B\) and \(E \land A = 0\) then we get \(E = 0\) a contradiction. Hence \(F\) is fuzzy *-connected.

Theorem 4.3: Let \(\{Y_\alpha : \alpha \in \Lambda\}\) be a collection of fuzzy *-connected sets in a fuzzy ideal topological space \((X, \tau, I)\). Then \(Y = \lor \{Y_\alpha : \alpha \in \Lambda\}\) is fuzzy *-connected provided there exists \(\beta \in \Lambda\) such that either (i) \(Y_\alpha\) and \(Y_\beta\) are not fuzzy *-separated for each \(\alpha \in \Lambda\), or (ii) \(Y_\alpha \land Y_\beta \neq 0\) for each \(\alpha \in \Lambda\).

Proof: Suppose \(Y\) is not fuzzy *-connected. Then \(Y = A \lor B\), where \(A\) and \(B\) are fuzzy *-separated sets in X. For an arbitrary \(\alpha \in \Lambda\), either \(a)\) \(Y_\alpha \leq A\) with \(Y_\alpha \land B = 0\) or \(b)\) \(Y_\alpha \leq B\) with \(Y_\alpha \land A = 0\), by Theorem 4.1. Similarly, \(c)\) \(Y_\beta = A\) with \(Y_\beta \land B = 0\) or \(d)\) \(Y_\beta = B\) with
$Y_\beta \land A = 0$. Without loss of generality we can assume that each 
$\{ Y_\alpha : \alpha \in \Lambda \}$ to be non-null, and hence exactly one of the possibilities 
(a) and (b), and exactly (c) and (d) will hold.

For case (ii), the possibilities (a) and (b) cannot happen and similarly 
(b) and (c) cannot hold simultaneously. For case (i), if (a) and (b) hold, 
then $Y_\alpha = Y_\alpha \land A$ and $Y_\beta = Y_\beta \land B$ are fuzzy *-separated, $A$ and $B$ are 
being so. This is a contradiction. Similarly for case (ii) the possibilities 
(b) and (c) together are to be ruled out.

Thus in any case, either $Y_\alpha \leq A$ with $Y_\alpha \land B = 0$ or $Y_\alpha \leq B$ with $Y_\alpha \land 
A = 0$ (but not both simultaneously) for each $\alpha \in \Lambda$. Now, $Y_\alpha \leq A$ and 
$Y_\alpha \leq B = 0$ and thus $B = 0$, a contradiction. Similarly, $Y_\alpha \leq B$ and 
$Y_\alpha \leq A = 0$ for all $\alpha \in \Lambda$ implies $A = 0$, again a contradiction.

**Theorem 4.4:** Let $Y$ be a fuzzy set of a fuzzy ideal topological space 
$(X, \tau, I)$ such that there exists at least one point $x \in X$, with $Y(x) > 1/2$. 
Then $Y$ is fuzzy *-connected if and only if two fuzzy points of $Y$ are 
contained in a fuzzy *-connected set contained in $Y$.

**Proof:** Necessity. Let $Y$ be fuzzy *-connected, then the condition is 
clearly true, irrespective of whether $Y(x) > 1/2$ at some point $x \in X$.

**Sufficiency.** Let $x \in X$ such that $Y(x) > 1/2$. For each $y \in Y_0 - \{x\}$ 
(where $Y_0$ denotes the support of $Y$) there exists a fuzzy *-connected 
set $A_y \leq Y$ such that $X_{Y(x)}, Y_{y(x)} \in A_y$. Clearly 
$\forall\{A_y : \ y \in Y_0 - \{x\}\} = Y$ and $\forall\{A_y : \ y \in Y_0 - \{x\}\}$ is fuzzy *-
connected by Theorem 4.2.

**Corollary 4.1:** A fuzzy ideal topological space $(X, \tau, I)$ is 
 fuzzy *-connected if and only if every pair of fuzzy points is contained 
in a fuzzy *-connected set.

**Theorem 4.5:** Let $f:(X, \tau, I) \rightarrow (Y, \sigma)$ be a fuzzy continuous surjection. If 
$E$ be fuzzy *-connected set in $X$. Then $f(E)$ is fuzzy connected set of $Y$.

**Proof:** Suppose that $f(E)$ is not fuzzy connected set of $Y$. Then there 
exist fuzzy separated sets $A$ and $B$ in $Y$ such that $f(E) = A \lor B$. 
Therefore there exist fuzzy open sets $U$ and $V$ such that $A \leq U$, $B \leq V$, 
$\lceil (A \lor U)$ and $\lceil (B \lor V)$. Now 
$E = f^{-1}(f(E)) = f^{-1}(A \lor B) = f^{-1}(A) \lor f^{-1}(B)$ 
and it can be easily verified that 
$f^{-1}(A) \leq f^{-1}(U)$; $f^{-1}(B) \leq f^{-1}(V)$, 
$\lceil (f^{-1}(A) \lor f^{-1}(U))$ and $\lceil (f^{-1}(V) \lor f^{-1}(B))$. Since $f$ is fuzzy 
continuous $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy open set in $X$. Since every
fuzzy open set is fuzzy *-open it follows that $f^{-1}(V)$ is fuzzy *-open set in $X$. Thus by Theorem 3.6, $f^{-1}(A)$ and $f^{-1}(B)$ are fuzzy *-separated in $X$. Hence $E$ is not fuzzy *-connected, which is a contradiction.

References


Received: April, 2012