Q-Fuzzy Interior Ideals in Semigroups

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Abstract

In this paper we investigate some properties of a Q-fuzzy interior ideal of a semigroup. We also consider Q-fuzzy characteristic interior ideals.

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1 Introduction

Let $S$ be a nonempty set. A fuzzy subset of $S$ is, by definition, an arbitrary mapping $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The important concept of fuzzy set put forth by Zadeh in 1965 [12] has opened up keep insights and applications in a wide rang of scientific fields. A theory of fuzzy sets on ordered semigroups has been recently developed [1-5]. In this paper we investigate some properties of Q-fuzzy interior ideals, and discuss Q-fuzzy characteristic interior ideals in semigroups.

2 Preliminary Notes

Let $Q$ be nonempty set. A function $f$ from $S \times Q$ to real closed interval $[0, 1]$ is called a Q-fuzzy subset of $S$. Let $A$ be anonempty subset of $S$. We denote by $f_A$ the characteristic mapping of $A$, that is, the mapping of $S \times Q$ into $[0, 1]$ define by

$$f_A(x, q) := \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A.
\end{cases}$$
Then $f_A$ is a $Q$-fuzzy subset of $S$.
For a $Q$-fuzzy set $f$ in a semigroup $S$ and $t \in [0,1]$, the set
\[ f_t = \{ x \in S \mid f_t(x, q) \geq t, \forall q \in Q \} \]
is called a level subset of $f$.

A $Q$-fuzzy set $f$ in a semigroup $S$ is called a $Q$-fuzzy subsemigroup of $S$ if $f(xy, q) \geq \min\{f(x, q), f(y, q)\}$ for all $x, y \in S, q \in Q$.

A subsemigroup $A$ of a semigroup $S$ is called an interior ideal of $S$ if $SAS \subseteq S$. A $Q$-fuzzy subsemigroup $f$ of a semigroup $S$ is called a $Q$-fuzzy interior ideal of $S$ if $f(xy, q) \geq f(a, q)$ for all $x, a, y \in S, q \in Q$.

3 Main Results

**Theorem 3.1** If $f$ is a $Q$-fuzzy interior ideal of a semigroup $S$, then the level subset $f_t$ of $f$ is an interior ideal of $S$ for every $t \in [0,1]$, when $f_t \neq \emptyset$.

**Proof.** Assume that $f_t \neq \emptyset$. Let $a, b \in f_t, q \in Q$ for every $t \in [0,1]$. Then $f(a, q) \geq t$ and $f(b, q) \geq t$, which imply that $f(ab, q) \geq \min\{f(a, q), f(b, q)\} \geq t$, so that $ab \in f_t$. Hence $f_t$ is a subsemigroup of $S$. Now let $x, y \in S, q \in Q$ and $a \in f_t$ for every $t \in [0,1]$. Then $f_t(xy, q) \geq f_t(a, q) \geq t$, which implies that $xay \in f_t$. Consequently $f_t$ is an interior ideal of $S$. \hfill \blacksquare

**Definition 3.2** Let $f$ be a $Q$-fuzzy interior ideal of a semigroup $S$. The interior ideals $f_t, t \in [0,1]$, are called level interior ideals of $f$.

**Theorem 3.3** Let $A$ be an interior ideal of a semigroup $S$. Then for every $t \in (0,1)$, there exists a $Q$-fuzzy interior ideal $f$ of $S$ such that $f_t = A$.

**Proof.** Let $A$ be an interior ideal of $S$ and let $f$ be a $Q$-fuzzy set in $S$ define by
\[ f(x, q) := \begin{cases} t & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \]
where $t$ is a fixed number in $(0,1)$. Let $x, y \in S, q \in Q$. If $x, y \in A$, then $xy \in A$. Hence $f(xy, q) = t = \min\{f(x, q), f(y, q)\}$. If $x, y \notin A$, then $f(x, q) = 0 = f(y, q)$, and so $f(xy, q) \geq \min\{f(x, q), f(y, q)\}$. If exactly one of $x$ and $y$ belongs to $A$ then exactly one of $f(x, q)$ and $f(y, q)$ is equal to 0. So $f(xy, q) \geq \min\{f(x, q), f(y, q)\}$. Hence $f$ is a $Q$-fuzzy subsemigroup of $S$. Now let $x, a, y$ be any elements of $S$. If $a \in A$ then $xay \in A$. Thus $f(xy, q) = t = f(a, q)$. If $a \notin A$ then $f(a, q) = 0$, and so $f(xy, q) \geq 0 = f(a, q)$. Therefore $f$ is a $Q$-fuzzy interior ideal of $S$, and clearly $f_t = A$. \hfill \blacksquare
**Theorem 3.4** Suppose \( f \) is a \( Q \)-fuzzy interior ideal of \( S \). Then two level interior ideals \( f_{t_1}, f_{t_2} \) (with \( t_1 < t_2 \)) of \( f \) are equal if and only if there is no \( x \in S \) such that \( t_1 \leq f(x, q) < t_2 \).

**Proof.** Assume that \( f_{t_1} = f_{t_2} \) for \( t_1 < t_2 \) and that there exists \( x \in S \) such that \( t_1 \leq f(x, q) < t_2 \) for all \( q \in Q \). Then \( f_{t_2} \) is a proper subset of \( f_{t_1} \). This is a contradiction.

Conversely suppose there is no \( x \in S \) and \( q \in Q \) such that \( t_1 < t_2 \) implies \( f_{t_2} \subseteq f_{t_1} \). If \( x \in f_{t_1} \), then \( f(x, q) \geq t_1 \). Since \( f(x, q) \not< t_2 \), we get \( f(x, q) \geq t_2 \) or \( x \in f_{t_2} \). Thus \( f_{t_1} = f_{t_2} \), completing the proof.

**Theorem 3.5** Let \( f \) be a \( Q \)-fuzzy interior ideal of semigroup \( S \). If \( Im(f) \) is finite, then the family of interior ideals \( f_t, t \in Im(f) \), constitutes all the level interior ideals of \( f \).

**Proof.** Suppose that \( Im(f) \) is finite. Since \( Im(f) \subseteq [0, 1] \), without loss of generality we may assume that \( Im(f) = \{t_1, t_2, \ldots, t_n\} \) where \( t_1 < t_2 < \cdots < t_n \). Let \( t \in [0, 1] \) and \( t \notin Im(f) \). If \( t < t_1 \) then \( f_t \subseteq f_{t_1} \). Since \( f_{t_1} = S \), we have that \( f_t = S \) and \( f_t = f_{t_1} \). If \( t_i < t < t_{i+1} \), \( 1 \leq i \leq n - 1 \), then there is no \( x \in S \) and \( q \in Q \) such that \( t \leq f(x, q) < t_{i+1} \). It follows from Theorem 3.4 that \( f_t = f_{t_{i+1}} \). Hence the level interior ideal \( f_t \) is in \( \{f_{t_i} \mid i = 1, 2, \ldots, n\} \). The proof is complete.

**Lemma 3.6** Let \( f \) be a \( Q \)-fuzzy interior ideal of a semigroup \( S \). If \( s \) and \( t \) belong to \( Im(f) \) such that \( f_s = f_t \) then \( s = t \).

**Proof.** Assume that \( s \neq t \), say \( s < t \). Then there is \( x \in S \) and \( q \in Q \) such that \( f(x, q) = s < t \), and so \( x \in f_s \) and \( x \notin f_t \). Thus \( f_s \neq f_t \), a contradiction. This completes the proof.

**Theorem 3.7** Let \( S \) be a semigroup and let \( g \) and \( f \) be two \( Q \)-fuzzy interior ideals of \( S \) with identical family of level interior ideals. If \( Im(g) = \{s_1, s_2, \ldots, s_n\} \) and \( Im(f) = \{t_1, t_2, \ldots, t_m\} \), where \( s_1 > s_2 > \cdots > s_n \) and \( t_1 > t_2 > \cdots > t_m \), then

(i) \( m = n \);
(ii) \( g_{s_k} = f_{t_k}, k = 1, 2, \ldots, n \);
(iii) if \( x \in S, q \in Q \) such that \( g(x, q) = s_k \) then \( f(x, q) = t_k \) for \( k = 1, 2, \ldots, n \).

**Proof.** Using Theorem 3.5 we have that the only level interior ideals of \( g \) and \( f \) are \( g_{s_k} \) and \( f_{t_k} \), respectively. Since \( g \) and \( f \) have the identical family of level interior ideals, it follows that \( n = m \), so that (i) holds.

To prove (ii), using Theorem 3.5 again we get \( \{g_{s_1}, g_{s_2}, \ldots, g_{s_n}\} = \{f_{t_1}, f_{t_2}, \ldots, f_{t_n}\} \), and by Theorem 3.4 we have \( g_{s_1} \subset g_{s_2} \subset \cdots \subset g_{s_n} = S \) and \( f_{t_1} \subset \).
Let \( f_{t_2} \subseteq \ldots \subseteq f_{t_n} = S \). Hence \( g_{s_k} = f_{t_k} \) for \( k = 1, 2, \ldots, n \) and (ii) holds. Now let \( x \in S, q \in Q \) be such that \( g(x, q) = s_k \) and let \( f(x, q) = t_j \). Noticing that \( x \in f_{t_k} \), i.e., \( f(x, q) \geq t_k \), we get \( t_j \geq t_k \). Thus \( f_{t_j} \subseteq f_{t_k} \). Since \( x \in f_{t_j} = g_{s_j} \), we obtain \( s_k = g(x, q) \geq s_j \). It follows that \( g_{s_k} \subseteq g_{s_j} \). By (ii), \( f_{t_k} = g_{s_k} \subseteq g_{s_j} = f_{t_j} \). Therefore \( f_{t_k} = f_{t_j} \), and by Lemma 3.6 we have \( t_k = t_j \). Hence \( f(x, q) = t_k \). The proof is complete.

**Theorem 3.8** Let \( g \) and \( f \) be two \( Q \)-fuzzy interior ideals of a finite semigroup \( S \) such that the families of level interior ideals of \( g \) and \( f \) are identical. Then \( g = f \) if and only if \( \text{Im}(g) = \text{Im}(f) \).

**Proof.** Necessity is obvious. Assume that \( \text{Im}(g) = \text{Im}(f) = \{t_1, \ldots, t_n\} \) where \( t_1 > t_2 > \cdots > t_n \). Let \( x_1, \ldots, x_n \) be distinct elements of \( S \) and \( q \in Q \) such that \( g(x_k, q) = t_k \) for \( 1 \leq k \leq n \). By Theorem 3.7(iii), \( f(x_k, q) = t_k \) for \( 1 \leq k \leq n \). Since for any \( x \in S \) there exists some \( t_k \) such that \( g(x, q) = t_k \), and so \( x \in g_{t_k} = f_{t_k} \). Hence \( f(x, q) \geq t_k \), it follows that \( f(x, q) \geq g(x, q) \). By the same argument, we get \( g(x, q) \geq f(x, q) \). Therefore \( g(x, q) = f(x, q) \), which shows that \( g = f \). This completes the proof.

Let \( S \) and \( T \) be semigroups. By a homomorphism we mean a mapping \( f : S \to T \) satisfying the identity \( f(xy) = f(x)f(y) \) for all \( x, y \in S \). Throughout, \( \text{Aut}(S) \) will denote the set of all automorphisms of \( S \).

**Definition 3.9** An interior ideal \( A \) of a semigroup \( S \) is called a characteristic interior ideal of \( S \) if \( f(A) = A \) for all \( f \in \text{Aut}(S) \).

**Definition 3.10** A \( Q \)-fuzzy interior ideal \( \mu \) of a semigroup \( S \) is called a \( Q \)-fuzzy characteristic interior ideal of \( S \) if \( \mu(f(x), q) = \mu(x, q) \) for all \( x \in S, q \in Q \) and all \( f \in \text{Aut}(S) \).

**Theorem 3.11** If \( \mu \) is a \( Q \)-fuzzy characteristic interior ideal of a semigroup \( S \), then each level interior ideal of \( S \) is a characteristic interior ideal of \( S \).

**Proof.** Let \( t \in \text{Im}(\mu), f \in \text{Aut}(S) \) and \( x \in \mu_{t}, q \in Q \). Since \( \mu \) is a \( Q \)-fuzzy characteristic interior ideal, we obtain \( \mu(f(x), q) = \mu(x, q) \geq t \). It follows that \( f(x) \in \mu_{t} \) and hence \( f(\mu_{t}) \subseteq \mu_{t} \). Now let \( x \in \mu_{t} \) and \( y \in S \) be such that \( f(y) = x \). Then \( \mu(y, q) = \mu(f(y), q) = \mu(x, q) \geq t \), whence \( y \in \mu_{t} \). It follows that \( x = f(y) \in f(\mu_{t}) \), so that \( \mu_{t} \subseteq f(\mu_{t}) \). Thus \( \mu_{t}, t \in \text{Im}(\mu), \) is a characteristic interior ideal of \( S \).

The following lemma is obvious and we omit the proof.
Lemma 3.12 Let $S$ be a semigroup and let $x \in S, q \in Q$. If $\mu$ is a $Q$-fuzzy interior ideal of $S$, then $\mu(x, q) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.

Theorem 3.13 Let $\mu$ be a $Q$-fuzzy interior ideal of a semigroup $S$. If each level interior ideal of $\mu$ is a characteristic interior ideal of $S$, then $\mu$ is a $Q$-fuzzy characteristic interior ideal of $S$.

Proof. Let $f \in \text{Aut}(S)$ and let $x \in S, q \in Q$ be such that $\mu(x, q) = t$. Then $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$, by Lemma 3.12. Since $f(\mu_t) = \mu_t$ by hypothesis, we get $f(x) \in \mu_t$ and hence $\mu(f(x), q) \geq t$. Let $s = \mu(f(x), q)$. If possible, let $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$. Since $f$ is one-one, it follows that $x \in \mu_s$, which is contradiction. Hence $\mu(f(x), q) = t = \mu(x, q)$, showing that $\mu$ is a $Q$-fuzzy characteristic interior ideal of $S$.

Lemma 3.14 If $\mu$ is a $Q$-fuzzy interior ideal of a semigroup $S$, then so is $\mu^\alpha$ for every real number $\alpha \geq 0$, where $\mu^\alpha$ is defined by $\mu^\alpha(x, q) = (\mu(x, q))^\alpha$ for all $x \in S, q \in Q$.

Proof. Let $x, a, y \in S, q \in Q$ and let $\alpha \geq 0$ be any real number. If $\mu(x, q) \leq \mu(y, q)$, then $\mu(xy, q) \geq \mu(x, q)$ and $\mu(y, q) \geq \mu(x, q)$. This implies that $\mu^\alpha(xy, q) \geq \min\{\mu^\alpha(x, q), \mu^\alpha(y, q)\}$. The argument is similar if $\mu(x, q) \geq \mu(y, q)$. Therefore $\mu^\alpha$ is a $Q$-fuzzy subsemigroup of $S$. Since $\mu(xay, q) \geq \mu(a, q)$, clearly $\mu^\alpha(xay, q) = (\mu(xay, q))^\alpha \geq (\mu(a))^\alpha = \mu^\alpha(a)$. This completes the proof.

References


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