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An Accurate Self-Starting Initial Value Solvers for Second Order Ordinary Differential Equations

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Abstract

This paper focuses on the development of continuous and discrete algorithms for the numerical solution of ordinary differential equations. A continuous hybrid two-step method is developed for second order initial value problems with Chebyshev polynomial as basis function through collocation and interpolation techniques. By selection of equally-spaced points for both interpolation and collocation, new efficient, self-starting and zero-stable finite difference method coupled as block method is produced which is generally more accurate when compared with existing methods. The method is analyzed, its order and error constants are determined with investigations made on consistency and stability.

Mathematics Subject Clarification: 65L05

Keyword: Hybrid, Collocation, Interpolation, Block Method

Introduction

The continuous integration algorithms for the numerical solution of the initial value problems (IVPs) have been discussed in the literature. The works of Lie and Norsett (1989) and Onumanyi et al (1994) focused on the construction of continuous multistep methods by employing the multistep collocation approach. The use of finite difference to construct continuous implicit schemes through which

the block formulae are derived has been discussed extensively by some scholars such as Awoyemi et al (2011), Olabode (2009), Adesanya et al (2009) to mention a few.
The traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation, Norsett (1989) and Onumanyi et al (1994). To obtain multiple discrete hybrid method, the continuous implicit hybrid method is evaluated at some selected points involving grid and off-grid points along with its first derivative. For the derivation of the block methods, the multiple discrete hybrid formulae obtained are solved simultaneously and the resulting equations constituted a block from which a number of explicit methods will be obtained.

Materials and Methods

We consider here the derivation of the proposed continuous hybrid two-step block methods. This we do by approximating the analytical solution of

$$y'' = f(x, y, y'), y'(a) = z_0, y(a) = y_0 \tag{1}$$

where f is a continuous function, with a Chebyshev polynomial in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j T_j(x) \tag{2}$$

on the partition $a = x_0 < x_1 < ... < x_n < x_{n+1} < ... < x_N = b$ of the integration interval [a, b], with a constant step size h, given by $h = x_{n+1} - x_n$; $n = 0, 1, ..., N - 1$. The second derivative of (2) is given by

$$y(x) = \sum_{j=0}^{r+s-1} a_j T_j^{''}(x) \tag{3}$$

where $x \in [a, b]$, the a_j's are real unknown parameters to be determined and r + s is the sum of the number of collocation and interpolation points.
We need to interpolate at at least two points to be able to approximate (2) and, to make this happen, we proceed by selecting two equally spaced offstep points. So, (2) is interpolated at $x = \frac{1}{2}$ & $x = \frac{3}{2}$ and its second derivative is collocated at x_{n+i}, i = 0, v and 1, so as to obtain a system of seven equations which are solved by Gaussian elimination method.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 2 & 7 & 26 & 97 & 362 & 1351 \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 \\ 0 & 0 & 16 & 0 & -64 & 0 & 144 \\ 0 & 0 & 16 & 96 & 320 & 800 & 1680 \\ 0 & 0 & 16 & 192 & 1472 & 9280 & 52368 \\ 0 & 0 & 16 & 288 & 3392 & 33120 & 290448 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \\ h^2 f_{n+\frac{3}{2}} \\ h^2 f_{n+2} \end{bmatrix} \quad (4)$$

Solving for a_j's and substituting the resulting equations into (2), we have

$$y(x) = \alpha_{\frac{1}{2}}(t) + \alpha_{\frac{3}{2}}(t) + \beta_0(t) + \beta_{\frac{1}{2}}(t) + \beta_1(t) + \beta_{\frac{3}{2}}(t) + \beta_2(t) \quad (5)$$

where

$$\left.\begin{aligned} \alpha_{\frac{1}{2}}(t) &= \frac{t}{2h} + 1 \\ \alpha_{\frac{3}{2}}(t) &= -\frac{t}{2h} \\ \beta_0(t) &= \frac{t^3}{96h^3} - \frac{t}{240h} + \frac{11t^4}{1152h^4} + \frac{t^5}{320h^5} + \frac{t^6}{2880h^6} \\ \beta_{\frac{1}{2}}(t) &= \frac{17t}{180h} + \frac{t^2}{8h^2} + \frac{5t^3}{144h^3} - \frac{5t^4}{288h^4} - \frac{t^5}{96h^5} - \frac{t^6}{720h^6} \\ \beta_1(t) &= \frac{19t}{120h} - \frac{t^3}{16h^3} + \frac{t^4}{192h^4} + \frac{t^5}{80h^5} + \frac{t^6}{480h^6} \\ \beta_{\frac{3}{2}}(t) &= \frac{t^3}{48h^3} + \frac{t^4}{288h^4} - \frac{t^5}{160h^5} - \frac{t^6}{720h^6} \\ \beta_2(t) &= \frac{t}{720h} - \frac{t^3}{288h^3} - \frac{t^4}{1152h^4} + \frac{t^5}{960h^5} + \frac{t^6}{2880h^6} \end{aligned}\right\} \quad (6)$$

and $t = h - 2x - 2x_k$.

Evaluating (5) at x_n, x_{n+1} and x_{n+2}, we have

$$\left.\begin{aligned}
y_n &= \frac{3}{2}y_{n+\frac{1}{2}} - \frac{1}{2}y_{n+\frac{3}{2}} + \frac{h^2}{1920}(37f_n + 432f_{n+\frac{1}{2}} + 222f_{n+1} + 32f_{n+\frac{3}{2}} - 3f_{n+2}) \\
y_{n+1} &= \frac{1}{2}y_{n+\frac{1}{2}} + \frac{1}{2}y_{n+\frac{3}{2}} + \frac{h^2}{1920}(f_n - 24f_{n+\frac{1}{2}} - 194f_{n+1} - 24f_{n+\frac{3}{2}} + f_{n+2}) \\
y_{n+2} &= \frac{3}{2}y_{n+\frac{3}{2}} - \frac{1}{2}y_{n+\frac{1}{2}} + \frac{h^2}{1920}(-3f_n + 32f_{n+\frac{1}{2}} + 222f_{n+1} + 432f_{n+\frac{3}{2}} + 37f_{n+2})
\end{aligned}\right\} \tag{7}$$

Differentiating (5) at $x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}$ and x_{n+2}, we have

$$y'(x) = \alpha'_{\frac{1}{2}}(t) + \alpha'_{\frac{3}{2}}(t) + \beta'_0(t) + \beta'_{\frac{1}{2}}(t) + \beta'_1(t) + \beta'_{\frac{3}{2}}(t) + \beta'_2(t) \tag{8}$$

where

$$\left.\begin{aligned}
\alpha'_{\frac{1}{2}}(t) &= \frac{1}{h} \\
\alpha'_{\frac{3}{2}}(t) &= \frac{1}{h} \\
\beta'_0(t) &= \frac{1}{120h} - \frac{t^2}{16h^3} - \frac{11t^3}{144h^4} - \frac{t^4}{32h^5} - \frac{t^5}{240h^6} \\
\beta'_{\frac{1}{2}}(t) &= \frac{5t^3}{36h^4} - \frac{t}{2h^2} - \frac{5t^2}{24h^3} - \frac{17}{90h} + \frac{5t^4}{48h^5} + \frac{t^5}{60h^6} \\
\beta'_1(t) &= \frac{3t^2}{8h^3} - \frac{19}{60h} - \frac{t^3}{24h^4} - \frac{t^4}{8h^5} - \frac{t^5}{40h^6} \\
\beta'_{\frac{3}{2}}(t) &= \frac{t^4}{16h^5} - \frac{t^3}{36h^4} - \frac{t^2}{8h^3} + \frac{t^5}{60h^6} \\
\beta'_2(t) &= \frac{t^2}{48h^3} - \frac{1}{360h} + \frac{t^3}{144h^4} - \frac{t^4}{96h^5} - \frac{t^5}{240h^6}
\end{aligned}\right\} \tag{9}$$

Evaluating (8) at x_n, $x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}$ and x_{n+2}, we have

$$\left.\begin{aligned}
hy'_n &= y_{n+\frac{3}{2}} - y_{n+\frac{1}{2}} + \frac{h^2}{1440}(-239f_n - 918f_{n+\frac{1}{2}} - 192f_{n+1} - 106f_{n+\frac{3}{2}} + 15f_{n+2}) \\
hy'_{n+\frac{1}{2}} &= y_{n+\frac{3}{2}} - y_{n+\frac{1}{2}} + \frac{h^2}{360}(3f_n - 68f_{n+\frac{1}{2}} - 114f_{n+1} + f_{n+2}) \\
hy'_{n+1} &= y_{n+\frac{3}{2}} - y_{n+\frac{1}{2}} + \frac{h^2}{1440}(-7f_n + 74f_{n+\frac{1}{2}} - 74f_{n+\frac{3}{2}} + 7f_{n+2}) \\
hy'_{n+\frac{3}{2}} &= y_{n+\frac{3}{2}} - y_{n+\frac{1}{2}} + \frac{h^2}{360}(f_n + 114f_{n+1} + 68f_{n+\frac{3}{2}} - 3f_{n+2}) \\
hy'_{n+2} &= y_{n+\frac{3}{2}} - y_{n+\frac{1}{2}} + \frac{h^2}{1440}(-15f_n + 106f_{n+\frac{1}{2}} + 192f_{n+1} + 918f_{n+\frac{3}{2}} + 239f_{n+2})
\end{aligned}\right\} \quad (10)$$

Combining and solving (7) and (10) simultaneously, explicit schemes (11) are obtained.

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{h^2}{5760}(367f_n + 540f_{n+\frac{1}{2}} - 282f_{n+1} + 116f_{n+\frac{3}{2}} - 21f_{n+2}) \\
y_{n+\frac{3}{2}} &= y_n + \frac{3}{2}hy'_n + \frac{h^2}{640}(147f_n + 468f_{n+\frac{1}{2}} + 54f_{n+1} + 60f_{n+\frac{3}{2}} - 9f_{n+2}) \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{360}(53f_n + 144f_{n+\frac{1}{2}} - 30f_{n+1} + 16f_{n+\frac{3}{2}} - 3f_{n+2}) \\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{45}(14f_n + 48f_{n+\frac{1}{2}} + 12f_{n+1} + 16f_{n+\frac{3}{2}}) \\
y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{1440}(251f_n + 646f_{n+\frac{1}{2}} - 264f_{n+1} + 106f_{n+\frac{3}{2}} - 19f_{n+2}) \\
y'_{n+1} &= y'_n + \frac{h}{180}(29f_n + 124f_{n+\frac{1}{2}} + 24f_{n+1} + 4f_{n+\frac{3}{2}} - f_{n+2}) \\
y'_{n+\frac{3}{2}} &= y'_n + \frac{h}{160}(27f_n + 102f_{n+\frac{1}{2}} + 72f_{n+1} + 42f_{n+\frac{3}{2}} - 3f_{n+2}) \\
y'_{n+2} &= y'_n + \frac{h}{45}(7f_n + 32f_{n+\frac{1}{2}} + 12f_{n+1} + 32f_{n+\frac{3}{2}} + 7f_{n+2})
\end{aligned}\right\} \quad (11)$$

Analysis of the Method

Here, the order, error constant and consistency of the method are discussed.

The explicit schemes (7) derived are discrete schemes belonging to the class of LMM of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} \tag{12}$$

Associated with (12) is the linear differential operator L defined by

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)] \tag{13}$$

Expanding (13) by Taylor series, we have

$$\mathrm{L[y(x);h]} = C_0 y(x) + C_1 h y'(x) + ... + C_q h^q y^{(q)}(x) + ... \tag{14}$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_k$$
$$C_1 = \alpha_1 + 2\alpha_2 + ... + k\alpha_k$$
$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + ... + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + ... + \beta_k)$$

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$$C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + ... + k^p\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-2}\beta_2 + ... + k^{q-2}\beta_k), q \geq 3$$

Definition 1
The LMM (12) is said to be of order p if C0 = $C_1 = C_2 = ... = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant, see Lambert (1973). According to this definition, the discrete schemes (7) have order p = $(5, 5, 5)^T$ with error constants

$$\left(\frac{-1}{30720}, 1.0011\times10^{-6} \text{ and } \frac{1}{30720}\right)^T.$$

Definition 2
The LMM (12) is said to be consistent if it is of order p ≥ 1 and its first and second characteristic polynomials defined as $\rho(z) = \sum_{j=0}^{k}\alpha_j z^j$ *and* $\sigma(z) = \sum_{j=0}^{k}\beta_j z^j$

where z satisfies

$$(i)\sum_{j=0}^{k}\alpha_j = 0, (ii)\ \rho(1) = \rho'(1) = 0, (iii)\ \rho''(1) = 2!\sigma(1), \textit{see Lambert}\ (1973).$$

The discrete schemes derived are all of order greater than one and satisfy the conditions (i) - (iii).

Definition 3
The LMM (12) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one has multiplicity not greater than two.
All the roots of the derived schemes have been verified to be less than or equal to 1 and $|z| = 1$, simple.
Numerical Examples
Here, we consider the application of the derived schemes to three test problems for the efficiency and accuracy of the method implemented as block method.

$\text{Pr}\,\textit{oblem}\,1$

$$y'' = y + xe^{3x}, y(0) = \frac{-3}{32}, y'(0) = \frac{-5}{32}, h = 0.0025$$

$$\textit{Exact Solution}: y(x) = \frac{4x-3}{32e^{-3x}}.$$

Source : *Adesanya et al* (2009).

$\text{Pr}\,\textit{oblem}\,2$

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.0025$$

$$\textit{Exact Solution}: y(x) = 1 + \frac{1}{2}\ln\frac{2+x}{2-x}$$

Source : *Awoyemi et al* (2012).

$\text{Pr}\,\textit{oblem}\,3$

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, y(1) = 1, y'(1) = 1, h = \frac{0.1}{32}$$

$$\textit{Exact Solution}: y(x) = \frac{5}{3x} - \frac{2}{3x^4}.$$

Source : *Yahaya and Badmus* (2009).

Table of Results

Table 1: The exact solutions, the computed results and the absolute errors from problems 1

X	Proposed Method	Exact Solutions	Absolute Errors
0.0025	-0.094140915761849	-0.094140915761849	0
0.0050	-0.094532404142339	-0.094532404142339	0
0.0075	-0.094924451608388	-0.094924451608388	1.665334536937735e-016
0.0100	-0.095317044390700	-0.095317044390700	1.665334536937735e-016
0.0125	-0.095710168480980	-0.095710168480981	3.191891195797325e-016
0.0150	-0.096103809629113	-0.096103809629113	2.914335439641036e-016
0.0175	-0.096497953340316	-0.096497953340316	3.747002708109903e-016
0.0200	-0.096892584872264	-0.096892584872264	3.608224830031759e-016
0.0225	-0.097287689232184	-0.097287689232184	6.938893903907228e-017
0.0250	-0.097683251173920	-0.097683251173920	6.938893903907228e-017

Table 2: The exact solutions, the computed results and the absolute errors from problems 2

X	Proposed Method	Exact Solutions	Absolute Errors
0.0025	1.001250000651042	1.001250000651042	0
0.0050	1.002500005208353	1.002500005208353	0
0.0075	1.003750017578273	1.003750017578273	0
0.0100	1.005000041667291	1.005000041667292	0
0.0125	1.006250081382116	1.006250081382116	2.220446049250313e-016
0.0150	1.007500140629746	1.007500140629746	2.220446049250313e-016
0.0175	1.008750223317550	1.008750223317551	2.220446049250313e-016
0.0200	1.010000333353335	1.010000333353335	2.220446049250313e-016
0.0225	1.011250474645419	1.011250474645419	2.220446049250313e-016
0.0250	1.012500651102709	1.012500651102709	2.220446049250313e-016

Table 3: The exact solutions, the computed results and the absolute errors from problems 3

X	Proposed Method	Exact Solutions	Absolute Errors
0.003125	1.003076525857696	1.003076525857696	0
0.00625	1.006057503083516	1.006057503083516	0
0.009375	1.008944995088837	1.008944995088837	2.220446049250313e-016
0.0125	1.011741018167988	1.011741018167989	1.110223024625157e-015
0.015625	1.014447542686413	1.014447542686414	6.661338147750939e-016
0.01875	1.017066494235672	1.017066494235673	1.332267629550188e-015
0.021875	1.019599754756287	1.019599754756288	1.332267629550188e-015
0.025	1.022049163629431	1.022049163629432	1.110223024625157e-015
0.028125	1.024416518738402	1.024416518738403	8.881784197001252e-016
0.03125	1.026703577500805	1.026703577500806	8.881784197001252e-016

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