Recursiveness in Hilbert Spaces and Application to Mixed $ARMA(p, q)$ Process

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Abstract

In this paper, the gap between the nonhomogeneous linear recurrence relation and the mixed $ARMA(p, q)$ process in Hilbert spaces is provided. We focus ourselves on the case when the $ARMA(p, q)$ process is of Fibonacci type. Notably, involving properties of Fibonacci sequences, we reach to establish the new formulations of some characteristics of the $ARMA(p, q)$ process. For purpose of illustration, some examples are explored.

Mathematics Subject Classification: Primary: 37H10, 40A99; Secondary: 37M10, 62M10, 65Q05

Keywords: ARMA process, Fibonacci sequences, Hilbert space, nonhomogeneous linear recurrence relation, stochastic process

1 introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space and consider the stochastic process $\{X_t; t \in \mathbb{Z}\}$ of a mixed $ARMA(p, q)$ process, satisfying the following difference equation,

$$X_{t+1} = a_0 X_t + \cdots + a_{p-1} X_{t-p+1} + C_{t+1}, \quad (1)$$

such that $C_{t+1} = A_{t+1} + b_0 A_t + \cdots + b_{q-1} A_{t-q+1}$, where $a_0, a_1, \dots, a_{p-1} \in \mathbb{R}$ are the auto-regressive parameters, $b_0, b_1, \dots, b_{q-1} \in \mathbb{R}$ are the auto-regressive moving average parameters and $\{A_t; t \in \mathbb{Z}\}$ is a stochastic process describing the white noise (see [4], for example). The mixed $ARMA(p, q)$ process are currently used in the methods of prediction and time series (for more details see [4], [5], for example). For discrete indices $t = n$ and $\omega \in \Omega$, we set $x_n = X_n(\omega)$ and $c_n = C_n(\omega)$. It ensues from Equation (1) that the sequence $\{x_n\}_{n \geq 0}$ satisfies the following nonhomogeneous linear recursive equation,

$$x_{n+1} = a_0 x_n + \cdots + a_{p-1} x_{n-p+1} + c_{n+1}, \quad \text{for every } n \geq p-1, \quad (2)$$

where $x_0, \dots, x_{p-1}$ are the initial data. Solutions of Equation (2) are studied in the literature when c_n is polynomial or factorial polynomial (see [2], [7], [8], [9], for example). Recently, two methods have been proposed, for studying solutions of Equation (2) in [2], [7]. More precisely, in [7] a matrix theoretical approach for solving (2) is developed for a general $\{c_n\}_{n \geq 0}$. Beside in [2], a linearization process is introduced, when $\{c_n\}_{n \geq 0}$ verifies a nonhomogeneous linear recursive relation. When $c_n = 0$, for every n , it was proved in [11] that solutions of the homogeneous part of (2) are,

$$x_n^{<h>} = \rho(n, p) S_0 + \rho(n-1, p) S_1 + \cdots + \rho(n-p+1, p) S_{p-1}, \quad (3)$$

for $n \geq p$, where $S_k = a_{p-1} x_k + \cdots + a_k x_{p-1}$ ($0 \leq k \leq p-1$) and

$$\rho(n, p) = \sum_{k_0+2k_1+\cdots+pk_{p-1}=n-p} \frac{(k_0 + \cdots + k_{p-1})!}{k_0! \cdots k_{p-1}!} a_0^{k_0} \cdots a_{p-1}^{k_{p-1}}, \quad (4)$$

with $\rho(p, p) = 1$ and $\rho(n, p) = 0$ if $n \leq p-1$.

On the other hand, the closed relationship between stochastic process and the recursive relation (2), has been explored in [10], [11]. Moreover, it was established that some probabilistic properties are related both to recursiveness and Expression (4) in [13], [14]. Particularly, it was shown that Expression (4) is also a special case of Philippou polynomials (see [1], [3]).

In this paper, we are interested in studying a theoretical approach of the mixed $ARMA(p, q)$ (stochastic) process defined by (1) and the nonhomogeneous linear recurrence relation (2) in a real Hilbert space. We extend the matrix method of [7] to the case of a real Hilbert space (Section 2). We apply this extension in order to obtain a general solution of (2). Using the extension of the linearization method of [2] to the case of real Hilbert spaces, we study the case of $ARMA(p, q)$ process of Fibonacci type defined by $C_{t+1} = A_{t+1} = b_0 A_t + b_1 A_{t-1} + \cdots + b_{q-1} A_{t-q+1}$ (Section 3). Properties concerning the computation of some characteristics of the $ARMA(p, q)$ model, via the approach of linear recurrence relations, are explored (Section 4). Final

remarks on the operator methods and recursiveness are discussed in the last section.

Throughout of this paper $\mathcal{H}$ will denote a real Hilbert space.

2 Equation (2) in a Hilbert space

In this section, we extend the fundamental results established in [2], [7] upon the matrix method and the linearization process for solving Equation (2). To this aim, we recall first the following known result established in [7].

Proposition 2.1 *Let $\{v_n\}_{n \geq 0}$ be a real Fibonacci sequence given by $v_{n+1} = a_0 v_n + \cdots + a_{p-1} v_{n-p+1}$ for $n \geq 0$, where $v_0, v_1, \dots, v_{p-1}$ are specified initial data. Let $A = (\alpha_{i,j})_{1 \leq i,j \leq p}$ be the matrix defined by $\alpha_{1j} = a_{j-1}$, for every $1 \leq j \leq p$, and $\alpha_{ij} = \delta_{i-1,j}$, for every $2 \leq i \leq p$, $1 \leq j \leq p$. Then, for $n \geq 1$, the entries $a_{is}^{(n)}$ ($0 \leq i, s \leq p-1$) of the matrix powers A^n are given by $a_{is}^{(n)} = v_{n-i}^{(s)}$, where $\{v_n^{(s)}\}_{n \geq 0}$ are p copies of $\{v_n\}_{n \geq 0}$, whose initial conditions are $v_j^{(s)} = \delta_{s,j}$ ($0 \leq j \leq p-1$, $0 \leq s \leq p-1$). Here, $\delta_{s,j}$ represents the Kronecker symbol.*

In view of (3), we infer that for every s ($0 \leq s \leq p-1$) the sequence $\{v_{n+1}^{(s)}\}_{n \geq 0}$ can be formulated as follows. For $0 \leq n \leq p-1$ we have $v_n^{(s)} = \delta_{n,s}$ and

$$v_n^{(s)} = a_{p-s+1} \rho(n, p) + a_{p-1} \rho(n-1, p) + \cdots + a_{p-1} \rho(n-s+1, p), \text{ for } n \geq p, \quad (5)$$

where the $\rho(n, p)$ are given by Expression (4).

Consider $\{T_n\}_{n \geq 0}$ and $\{C_n\}_{n \geq 0}$ two sequences of $\mathcal{H}$ satisfying (2). For $n \geq p-1$, we set $Z_n = {}^t(T_n, \dots, T_{n-p+1}) \in \mathcal{H}^p$ and $D_n = {}^t(C_n, 0, \dots, 0) \in \mathcal{H}^p$, where tZ means the transpose of Z . It's easy to show that (2) is equivalent to the matrix equation $Z_{n+1} = AZ_n + D_{n+1}$, for $n \geq p-1$, where $A = (\alpha_{i,j})_{1 \leq i,j \leq p}$ is the matrix exhibited in Proposition 2.1. Thus, we obtain $Z_n = A^{n-p+1}Z_{p-1} + \sum_{k=p}^n A^{n-k}D_k$, for every $n \geq p$. Hence, similarly to the real case (see [2], [7]), we establish that the solution of Equation (2) is,

$$T_n = \sum_{s=0}^{p-1} v_{n-p+1}^{(s)} T_{p-s-1} + \sum_{k=p}^n v_{n-k}^{(0)} C_k, \text{ for every } n \geq 0. \quad (6)$$

The sequence $\{T_n^{<h>}\}_{n \geq 0}$ defined by $T_n^{<h>} = \sum_{s=0}^{p-1} v_{n-p+1}^{(s)} T_{p-s-1}$ is a solution of the homogeneous part of Equation (2). In addition, a particular solution $\{T_n^{<p>}\}_{n \geq 0}$ of (2) is $T_n^{<p>} = \sum_{k=p}^n v_{n-k}^{(0)} C_k = - \sum_{s=0}^{p-1} v_{n-p+1}^{(s)} T_{p-s-1} + T_n$, where

$T_n^{<p>} = 0$ for $n = 0, 1, \dots, p-1$. The sequence $\{T_n^{<p>}\}_{n \geq 0}$ is called the *fundamental particular solution* of (2). Thence, in light of Proposition 2.1 and Equation (6), we get the following general result,

Theorem 2.2 *Let $\{T_n\}_{n \geq 0}$ be a solution of (2) in a Hilbert space $\mathcal{H}$. Then, for $n \geq 0$, we have*

$$T_n = T_n^{<h>} + T_n^{<p>} = T_n^{<h>} - \sum_{s=0}^{p-1} v_{n-p+1}^{(s)} T_{p-s-1}^{<p>} + T_n^{<p>}, \quad (7)$$

where $\{T_n^{<p>}\}_{n \geq 0}$ is the fundamental particular solution of (2) and $\{T_n^{<h>}\}_{n \geq 0}$ is a solution of the homogeneous part of (2) with initial data $T_0, \dots, T_{p-1}$.

Now let consider the case when $\{C_n\}_{n \geq 0}$ is a linear recursive sequence. Set $\{W_n\}_{n \geq 0}$ a recursive sequence in $\mathcal{H}$, whose initial data are $W_0, \dots, W_{s-1}$ and $W_{n+1} = b_0 W_n + \dots + b_{s-1} W_{n-s+1}$, for $n \geq s-1$, where $b_0, \dots, b_{s-1}$ are fixed in $\mathbb{R}$. Suppose that $C_n = W_n$ then, for every n ($n \geq p+s-1$), Equation (2) yields $W_{n-j} = T_{n-j} - \sum_{k=0}^{p-1} a_k T_{n-k-j-1}$, for $0 \leq j \leq s-1$. And the substitution of these expressions in (2), implies that T_n ($n \geq p+s-1$) satisfies the following linear recurrence equation of order $p+s$,

$$T_{n+1} = \sum_{j=0}^{p-1} a_j T_{n-j} + \sum_{j=0}^{s-1} b_j T_{n-j} - \sum_{j=0}^{s-1} \sum_{k=0}^{p-1} b_j a_k T_{n-j-k-1}. \quad (8)$$

As a result, we get the following extension of the Linearisation Process of [2] (see Theorem 2.1 of [2]),

Theorem 2.3 (*Linearization Process*). *Let $\{T_n\}_{n \geq 0}$ be a sequence of $\mathcal{H}$ solution of (2) and suppose that $C_n = W_n$, where $\{W_n\}_{n \geq 0}$ is a homogeneous linear recursive sequence of order s . Then, the sequence $\{T_n\}_{n \geq 0}$ satisfies a homogeneous linear recursive equation of order $m = r+s$ in $\mathcal{H}$, with initial data $T_0, \dots, T_{p+s-1}$ and whose coefficients are those of the polynomial $p(z) = p_1(z)p_2(z)$, where $p_1(z) = z^p - \sum_{j=0}^{p-1} a_j z^{p-j-1}$ and $p_2(z) = z^s - \sum_{j=0}^{s-1} b_j z^{s-j-1}$.*

Remark 2.4 *For $E = \mathbb{R}$, it is well known that the solution of (2) can be written under the form $T_n = T_n^{<h>} + T_n^{<p>}$, where $\{T_n^{<h>}\}_{n \geq 0}$ and $\{T_n^{<p>}\}_{n \geq 0}$ are the solutions of the homogeneous part and the particular solution of (2) respectively. It is easy to verify that this property is still valid for sequences (2) in $\mathcal{H}$. Following the similar method of [11], we can bring out that the particular solution of (2) is $T_n^{<p>} = T_n - T_n^{<h>}$, for $n \geq 0$.*

3 Study of the ARMA(p, q) process equations

3.1 General setting.

In this section we are concerned by the Hilbert space $\mathcal{H} = L^2(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A}, P)$ is a probability space. That is, $\mathcal{H}$ is the collection of the real-valued random variables (*r.r.v.* for short) X with finite variance, viz.,

$$E(X^2) = \int X^2(\omega)P(d\omega) = \int X^2 dP < +\infty.$$

Here $\mathcal{H}$ is equipped with the usual inner product $\langle X, Y \rangle = E(XY) = \int XY dP$. Expressions (5)-(6) and Theorem 2.2, allow us to formulate the result.

Theorem 3.1 *Let $\{X_t \in \mathcal{H}; t \in \mathbb{Z}\}$ be an ARMA(p, q) process and $\{A_t \in \mathcal{H}; t \in \mathbb{Z}\}$ its associated white noise. Then, for every $t \geq p$, we have*

$$X_t = \rho(t - p + 1, p)X_{p-1} + [a_{p-2}\rho(t, p) + a_{p-1}\rho(t - 1, p)]X_{p-2} + \cdots + \quad (9)$$

$$\left[\sum_{j=0}^s a_{p-s+j-1}\rho(t - j, p) \right] X_{p-s+1} + \cdots + \left[\sum_{j=0}^{p-1} a_j \rho(t - j, p) \right] X_0 + \sum_{k=p}^t \rho(t - k, p) C_k,$$

where $X_0, X_1, \dots, X_{p-1}$ are the *r.r.v.* of the initial data, the $\rho(n, p)$ are given by (4) and $C_t = A_t + b_0 A_{t-1} + \cdots + b_{q-1} A_{t-q}$.

Similarly, for $t \leq 0$, we set $Y_{p-t-1} = X_t$ and $b_0 = \frac{-a_0}{a_{p-1}}, \dots, b_j = \frac{-a_{p-j-1}}{a_{p-1}}, \dots, b_{p-2} = \frac{-a_{p-2}}{a_{p-1}}, b_{p-1} = \frac{1}{a_{p-1}}$. Thus, we have

$$X_t = \Delta_0 X_0 + \Delta_1 X_1 + \cdots + \Delta_{s-1} X_{s-2} + \cdots + \Delta_{p-1} X_{p-1} + \Gamma(-t), \quad (10)$$

where $\Delta_0 = \rho(t - p + 1, p)$, $\Delta_1 = b_{p-2}\hat{\rho}(-t + p - 1, p) + b_{p-1}\hat{\rho}(-t + p - 2, p)$, $\Delta_{s-1} = \sum_{j=0}^s b_{p-s+j-1}\hat{\rho}(-t + p - 1 - j, p)$, $\Delta_{p-1} = \sum_{j=0}^{p-1} b_j \hat{\rho}(-t + p - 1 - j, p)$ and $\Gamma(-t) = \sum_{k=p}^{-t} \hat{\rho}(-t + p - 1 - k, p) \left[\frac{-C_{k+2p-1}}{a_{p-1}} \right]$, with

$$\hat{\rho}(n, p) = \sum_{k_0+2k_1+\cdots+pk_{p-1}=n-p} \frac{(k_0 + \cdots + k_{p-1})!}{k_0! \cdots k_{p-1}!} b_0^{k_0} \cdots b_{p-1}^{k_{p-1}}.$$

The combinatorial expressions (9)-(10) show that every *r.r.v.* X_t (for $t \geq p$) can be expressed in terms of the initial *r.r.v.* $X_0, X_1, \dots, X_{p-1}$ and A_t ($t \in \mathbb{Z}$) the *r.r.v.* of the white noise. As far as we know (9)-(10) are not current in the literature on the ARMA(p, q) models.

EXAMPLE - Study of the ARMA(2, q) process. Consider $\{X_t\}_{t \in \mathbb{Z}}$ an ARMA(2, q) process defined by the random vector of initial data (X_0, X_1) , and

$$X_t = a_0 X_{t-1} + a_1 X_{t-2} + C_t, \quad (11)$$

where $C_{t+1} = A_{t+1} + b_1 A_t + \dots + b_q A_{t-q+1}$. Let $P(z) = z^2 - a_0 z - a_1 = (z - \lambda_1)(z - \lambda_2)$ be the characteristic polynomial of the homogeneous part of this process, where $\lambda_1 \neq 0, \lambda_2 \neq 0$ are in $\mathbb{R}$. For studying the process (11), we discuss the two cases $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$.

For $\lambda_1 = \lambda_2$, a direct computation shows that the Binet formula of $\rho(t, 2)$ takes the form $\rho(t, 2) = \lambda^{t-2}(-1 + t)$. Thereby, for every $t \geq 2$, we have

$$X_t = (t-2)\lambda^{t-3}X_1 + (2\lambda^{t-1}(-1+t) - (t-2)\lambda^{t-1})X_0 + \sum_{k=2}^t \lambda^{t-k-2}(t-k-1)C_k.$$

In the same way, for $t \leq 0$ we obtain $\hat{\rho}(-t, 2) = (\frac{1}{\lambda})^{-t-2}(-1-t)$ and thus

$$X_t = (\frac{1}{\lambda})^{-t-1}X_0 + [2(-1)^{-t}(\frac{1}{\lambda})^{-t}(-t) + (-1)^{-t}(\frac{1}{\lambda})^{-t}(-1-t)]X_1 + \Gamma(t),$$

where $\Gamma(t) = \sum_{k=2}^{1-t} \lambda^{-t-k+1}(-t-k)C_{3-k}$.

For $\lambda_1 \neq \lambda_2$, a similar calculation leads to the Binet formula of $\rho(n, 2)$ is $\rho(n, 2) = \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2}$. Hence, for $t > 0$, we get,

$$X_t = \frac{\lambda_1^{t-2} - \lambda_2^{t-2}}{\lambda_1 - \lambda_2}X_1 + [(\lambda_1 + \lambda_2)(\frac{\lambda_1^{t-1} - \lambda_2^{t-1}}{\lambda_1 - \lambda_2}) + \lambda_1\lambda_2\frac{\lambda_1^{t-2} - \lambda_2^{t-2}}{\lambda_1 - \lambda_2}]X_0 + \Gamma(t),$$

where $\Gamma(t) = \sum_{k=2}^t \frac{\lambda_1^{t-k-1} - \lambda_2^{t-k-1}}{\lambda_1 - \lambda_2}C_k$. For $t \leq 0$, we have

$$\hat{\rho}(-t, 2) = (\frac{1}{(\lambda_1\lambda_2)^{-t-2}}) \frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}.$$

Whence,

$$X_t = \Lambda_1(t)X_0 + \Lambda_2(t)X_1 + \Gamma(t),$$

where $\Gamma(t) = \sum_{k=2}^{1-t} \frac{\lambda_2^{-t-k} - \lambda_1^{-t-k}}{\lambda_2 - \lambda_1} (\frac{-1}{(\lambda_1\lambda_2)^{-t-k}})C_{3-k}$, $\Lambda_1(t) = \frac{1}{(\lambda_1\lambda_2)^{-t-2}} [\frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}]$, and $\Lambda_2(t, \lambda_j) = \frac{\lambda_1 + \lambda_2}{(\lambda_1\lambda_2)^{-t}} [\frac{\lambda_2^{-t} - \lambda_1^{-t}}{\lambda_2 - \lambda_1}] + (\frac{-1}{(\lambda_1\lambda_2)^{-t-1}}) (\frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1})$.

3.2 The ARMA(p, q) process of Fibonacci type and linearization process.

We say that an ARMA(p, q) process is of Fibonacci type if in (1) the C_{t+1} takes the form $C_{t+1} = A_{t+1} = b_0 A_t + \dots + b_{q-1} A_{t-q+1}$, where $\{A_t\}_{t \in \mathbb{Z}}$ is the

r.r.v. of white noise. That is to say, $\{A_t\}_{t \in \mathbb{Z}}$ satisfies a linear recursive relation of order q and with initial data $A_0, \dots, A_{q-1}$. With the aid of Theorem 2.2 and (8), the process $\{X_t\}_{t \in \mathbb{Z}}$ satisfies the following linear recursive equations of order $r = p + q$,

$$X_{t+1} = (a_0 + b_0)X_t + \sum_{j=0}^{r_1-1} (a_j + b_j - c_j)X_{t-j} + \sum_{j=r_1}^{r_2-1} v_j X_{t-j} - \sum_{j=r_2}^{p+q-1} c_j X_{t-j}, \quad (12)$$

where $c_j = \sum_{k+s=j; k \geq 1, s \geq 0} b_{k-1} a_s$ and $r_1 = \min(p, q)$, $r_2 = \max(p, q)$ with $v_j = a_j - c_j$ for $p > q$, $v_j = b_j - c_j$ for $p < q$ and $v_j = 0$ for $p = q$.

Theorem 3.2 *Let $\{X_t \in \mathcal{H}; t \in \mathbb{Z}\}$ be an ARMA(p, q) process of Fibonacci type and $\{A_t \in \mathcal{H}; t \in \mathbb{Z}\}$ its associated white noise, with initial data $X_0, \dots, X_{p-1}$ and $A_0, \dots, A_{q-1}$. Then, the process $\{X_t\}_{t \in \mathbb{Z}}$ satisfies a homogeneous linear recursive of order $r = p + q$ in $\mathcal{H}$, whose coefficients are those of the polynomial $p(z) = p_1(z)p_2(z)$, where $p_1(z) = z^p - \sum_{j=0}^{p-1} a_j z^{p-j-1}$ and $p_2(z) = z^q - \sum_{j=0}^{q-1} b_j z^{q-j-1}$.*

For reason of simplicity we suppose that $p \geq q$, since the same argumentation still valid in the case of $p \leq q$. It's easy to see that Expression (12) takes the linear form $X_{t+1} = \sum_{j=0}^{p+q-1} w_j X_{t-j}$, such that the coefficients w_j , derived from Theorem 2.3, are given by

$$\begin{aligned} w_0 &= a_0 + b_0, \quad w_j = a_j + b_j - c_j \text{ for } 1 \leq j \leq q-1, \quad w_j = a_j - c_j \text{ for } q \leq j \leq p-1 \\ &\quad \text{and } w_j = c_j \text{ for } p \leq j \leq p+q-1, \end{aligned} \quad (13)$$

and its initial data are

$$R_j = X_j \text{ for } 0 \leq j \leq p-1 \text{ and } R_j = X_j + A_j \text{ for } p \leq j \leq p+q-1. \quad (14)$$

Theorems 2.3, 3.2 and (3) yield,

Theorem 3.3 *Under the data of Theorem 3.2, the system $\{X_t; t \in \mathbb{Z}\}$ obey to a linear recursive equation of order $r = p + q$, whose coefficients are given by (13) and its initial data of r.r.v. by (14). Moreover, for every $t \geq p + q$, we have $X_t = \rho(t, p+q)S_0 + \rho(t-1, p+q)S_1 + \dots + \rho(t-p-q+1, p+q)S_{p+q-1}$, where $S_k = w_{p+q-1}X_k + \dots + w_k X_{p+q-1}$ ($0 \leq k \leq p+q-1$) and as in (4) the $\rho(n, p+q)$ are ,*

$$\rho(n, p+q) = \sum_{k_0+2k_1+\dots+(p+q)k_{p+q-1}=t-p-q} \frac{(k_0 + \dots + k_{p+q-1})!}{k_0! \dots k_{p+q-1}!} w_0^{k_0} \dots w_{p+q-1}^{k_{p+q-1}},$$

with $\rho(n, n) = 1$ and $\rho(n, s) = 0$ if $n \leq s-1$.

Similarly, for $t \leq 0$, by setting $Y_{p+q-t-1} = X_t$ and $\hat{w}_0 = \frac{-w_0}{w_{p+q-1}}, \dots, \hat{w}_j = \frac{-w_{p-j-1}}{w_{p+q-1}}, \dots, \hat{w}_{p+q-2} = \frac{-w_{p+q-2}}{w_{p+q-1}}, \hat{w}_{p+q-1} = \frac{1}{w_{p+q-1}}$, we obtain

$$X_t = \rho(p+q-t-1, p+q)\hat{S}_0 + \hat{\rho}(p+q-t-2, p+q)\hat{S}_1 + \dots + \hat{\rho}(-t, p+q)\hat{S}_{p+q-1},$$

where $\hat{S}_k = \hat{w}_{p+q-1}X_{p+q-k-1} + \dots + \hat{w}_kX_0$ ($0 \leq k \leq p+q-1$) and

$$\hat{\rho}(n, p+q) = \sum_{k_0+2k_1+\dots+(p+q)k_{p+q-1}=t-p-q} \frac{(k_0 + \dots + k_{p+q-1})!}{k_0! \dots k_{p+q-1}!} \hat{w}_0^{k_0} \dots \hat{w}_{p+q-1}^{k_{p+q-1}},$$

with $\hat{\rho}(n, n) = 1$ and $\hat{\rho}(n, s) = 0$ if $n \leq s-1$

EXAMPLE - Consider an $ARMA(2, 2)$ process $\{X_t; t \in \mathbb{Z}\}$ such that $P_1(z) = (z - \lambda_0)(z - \lambda_1)$ and $P_2(z) = (z - \lambda_2)(z - \lambda_3)$, where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are real numbers satisfying $\lambda_j \neq \lambda_k$ for $j \neq k$. Hence, we have $P(z) = (z - \lambda_0)(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)$. Then, for every $n \geq 0$, the Binet formula of $\rho(n, r)$ is $\rho(n, r) = \Gamma_n(\lambda_j) + \Theta_n(\lambda_j)$, where $\Gamma_n(\lambda_j) = \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_0^{n-1} - (\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_1^{n-1}}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$ and $\Theta_n(\lambda_j) = \frac{(\lambda_1 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)\lambda_2^{n-1} - (\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)\lambda_3^{n-1}}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$. Therefore, for $t \geq 3$, we obtain,

$$X_t = \rho(t, 4)S_0 + \rho(t-1, 4)S_1 + \rho(t-2, 4)S_2 + \rho(t-3, 4)S_3,$$

with $S_0 = \omega_3X_0 + \omega_2X_1 + \omega_1X_2 + \omega_0X_3$, $S_1 = \omega_3X_1 + \omega_2X_2 + \omega_1X_3$, $S_2 = \omega_3X_2 + \omega_2X_3$, $S_3 = \omega_3X_3$, where $\omega_0 = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$, $\omega_1 = \prod_{i \neq j, i, j=1}^4 \lambda_i \lambda_j$, $\omega_2 = \prod_{i \neq j \neq k, i, j, k=1}^4 \lambda_i \lambda_j \lambda_k$ and $\omega_3 = \lambda_0 \lambda_1 \lambda_2 \lambda_3$.

4 More on some characteristics of an $ARMA(p, q)$ process

In this section we are also concerned by the Hilbert space $\mathcal{H} = L^2(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A}, P)$ is a probability space. Consider an $ARMA(p, q)$ process (1), whose white noise A_t ($t \in \mathbb{Z}$) is with zero mean $E(A_t) = 0$ and constant variance $var(A_t) = \sigma^2$. Thus the sequence of moments $\{E(X_t)\}_{t \in \mathbb{Z}}$ satisfies the recursive equation $E[X_{t+1}] = a_0E[X_t] + \dots + a_{p-1}E[X_{t-p+1}]$. The Fibonacci sequence's properties, specially the Binet Formula, permits us to get.

Proposition 4.1 *Under the preceding data, suppose that $\lambda_1, \lambda_2, \dots, \lambda_r$ are the distinct roots of the polynomial $P(z) = z^p - a_0z^{p-1} - \dots - a_{p-1}$, of multiplicities $m_1, \dots, m_p$ (respectively). Then, we have $E[X_t] = \sum_{i=1}^r P_i(t)\lambda_i^t$ (Binet formula), for every $t \geq 0$, where each $P_i(z)$ is a polynomial of degree $m_i - 1$ ($1 \leq i \leq r$).*

Following (3)-(4), we manage to have the combinatorial form of $E[X_t]$.

Proposition 4.2 *Under the data of Proposition 4.1, we have*

$$E[X_t] = \Delta_{p-1}(t)E[X_{p-1}] + \cdots + \Delta_{p-s+1}(t)E[X_{p-1s+}] + \cdots + \Delta_0(t)E[X_0].$$

for every $t \geq p$, where $\Delta_{p-1}(t) = \rho(t-p+1, p)$, $\Delta_{p-s+1}(t) = \sum_{j=0}^s a_{p-s+j-1}\rho(t-j, p)$ and $\Delta_0(t) = \sum_{j=0}^{p-1} a_j\rho(t-j, p)$

Furthermore, we study others characteristics of X_t relying on the precedent results. We begin by the simple case when the $\{X_t; t \in \mathbb{Z}\}$ are mutually independent, the A_t are also mutually independent and X_{t-j} is independent of A_t for $j, t \geq 0$. Thus, we have $\text{var}[X_{t+1}] = a_0^2\text{var}[X_t] + \cdots + a_{p+1}^2\text{var}[X_{t-p+1}] + (1 + b_0 + \cdots + b_{q-1})\sigma^2$, where $t \in \mathbb{Z}$. Therefore, the sequence $\{\text{var}[X_t]\}_{t \geq 0}$ is a sequence (2) of characteristic polynomial $Q(z) = (z-1)(z^p - a_0^2z^{p-1} - \cdots - a_{p-1}^2)$, whose roots are $1, \lambda_1^2, \lambda_2^2, \dots, \lambda_r^2$, with multiplicities are $1, m_1, \dots, m_p$ (respectively). A direct application of the Binet formulas leads to derive the following result.

Proposition 4.3 *Under the data of Proposition 4.1, suppose that the X_t are mutually independent, the A_t mutually independent and X_{t-i} is independent of A_t for $t \geq 0$. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the distinct roots of the polynomial $P(z) = z^p - a_0z^{p-1} - \cdots - a_{p-1}$, of multiplicities $m_1, \dots, m_p$ (respectively). Then, we have $\text{var}[X_t] = \sum_{i=1}^r P_i(t)\lambda_i^{2t} + K$, where $K \in \mathbb{R}$ and each $P_i(z)$ ($1 \leq i \leq r$) is a polynomial of degree $m_i - 1$, whose coefficients are arisen from the initial conditions $\text{var}[X_0], \dots, \text{var}[X_{p-1}], \text{var}[X_p]$, either K .*

Now, we are going to discuss the general case without setting the independence condition of $\{X_t; t \in \mathbb{Z}\}$. To this aim, we release the combinatorial expression of $\text{var}[X_t]$ from Proposition 4.1, thereby we manage to have the following result.

Proposition 4.4 *Under the data of Proposition 4.1, suppose that the A_t are mutually independent and X_{t-i} is independent of A_t for $t \geq 0$. Then, for every $t \geq p$, we have*

$$\text{var}[X_t] = \sum_{i=0}^{p-1} \left(\sum_{j=0}^{p-1} \rho(t-j, p) \right)^2 \text{var}[X_i^2] + 2 \sum_{i < j, (i=1, j=2)}^{i=p-1, j=p} \alpha_{ij} \text{cov}(X_{p-i}, X_{p-j}),$$

where $\alpha_{ij} = \left(\sum_{k=0}^i a_{p-i+k-1}\rho(t-k, p) \right) \left(\sum_{h=0}^j a_{p-j+h-1}\rho(t-h, p) \right)$.

When the r.r.v. X_t ($t \in \mathbb{Z}$) are mutually independent, we have the following corollary.

Corollary 4.5 *Under the hypothesis of Proposition 4.4, suppose that $\{X_t; t \in \mathbb{Z}\}$ are mutually independent. Then, we have*

$$\text{var}[X_t] = \sum_{i=0}^{p-1} \left(\sum_{j=0}^{p-1} \rho(t-j, p) \right)^2 \text{var}[X_i^2].$$

We point out that we present, in Corollary 4.5, another expression of $\text{var}[X_t]$.

5 Operator methods and recursiveness

In the real case, difference equations of Fibonacci type are also related to operator methods for studying the $ARMA(p, q)$ process. That is, the equation $\phi(B)X_t = \theta(B)A_t$, where B is the auto-regressive operator, shows that $X_t = \psi(B)A_t$ and $A_t = \Pi(B)X_t$, where $\psi(B) = \phi(B)^{-1}\theta(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j$ and $\Pi(B) = \theta(B)^{-1}\phi(B) = 1 + \sum_{j=1}^{\infty} \pi_j B^j$ such that $\sum_{j=1}^{\infty} \psi_j^2 < +\infty$ and $\sum_{j=1}^{\infty} \pi_j^2 < +\infty$. For computing the ψ_j , in terms of the a_j ($0 \leq j \leq p-1$) and b_j ($0 \leq j \leq q-1$), we use the equation $\phi(B)\psi(B) = \theta(B)$. A straightforward computation shows that $\psi_1 = a_0 - b_0$, $\psi_j = \sum_{k=0}^{j-1} a_k \psi_{j-k+1} + a_{j-1} - b_{j-1}$ ($1 < j < p-1$), $\psi_p = \sum_{k=0}^{p-1} a_k \psi_{p-k+1} + a_{p-1} - b_{p-1} \epsilon_{p,q}$ (with $\epsilon_{p,q} = 1$ for $p \leq q$ and $\epsilon_{p,q} = 0$ otherwise), and

$$\psi_{n+1} = a_0 \psi_n + \cdots + a_{p-1} \psi_{n-p+1}, \quad \text{for } n \geq \max(p, q) - 1. \quad (15)$$

Similarly, for exhibiting the π_j in terms of the a_j ($0 \leq j \leq p-1$) and b_i ($0 \leq i \leq q-1$), a direct computation implies that $\pi_1 = a_0 - b_0$, $\pi_j = \sum_{k=0}^{j-1} b_k \pi_{j-k+1} + a_{j-1} - b_{j-1}$ ($1 < j < q-1$), $\pi_q = \sum_{k=0}^{p-1} a_k \psi_{p-k+1} + \epsilon_{p,q} a_{p-1} - b_{p-1}$ (with $\epsilon_{p,q} = 1$ for $p \leq q$ and $\epsilon_{p,q} = 0$ otherwise), and

$$\pi_{n+1} = b_0 \pi_n + \cdots + b_{q-1} \pi_{n-p+1}, \quad \text{for } n \geq \max(p, q) - 1. \quad (16)$$

The characteristic polynomial of (15) and (16) are $P_1(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$ and $P_2(z) = z^q - b_0 z^{q-1} - \cdots - b_{q-1}$. It's easy to see that $P_1(z) = z^p \phi(1/z)$ and $P_2(z) = z^q \theta(1/z)$. Therefore, the stationary and invertible $ARMA(p, q)$ process can be described as follows.

Proposition 5.1 *Let $\{X_t; t \in \mathbb{Z}\}$ be an $ARMA(p, q)$ process, with associated white noise is $\{A_t; t \in \mathbb{Z}\}$. Then, the following assertions are equivalent:*

- (i) $\{X_t; t \in \mathbb{Z}\}$ is stationary (respectively invertible),
- (ii) For every root λ of $P_1(z)$ (respectively $P_2(z)$) we have $|\lambda| < 1$.

Moreover, we have $\lim_{n \rightarrow +\infty} \psi_n = 0$ and $\lim_{n \rightarrow +\infty} \pi_n = 0$.

The last affirmation of Proposition 5.1 is an immediate consequence of the Binet formula of (15) and (16) (see [6], [9], for example). Moreover, if we

suppose that a_j ($0 \leq j \leq p-1$) and b_j ($0 \leq j \leq q-1$) are nonnegative, we can derive the asymptotic behavior of the two sequences $\{\psi_n\}_{n \geq 0}$ and $\{\pi_n\}_{n \geq 0}$, by using results of [6], [11].

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Received: June 5, 2014; Published: December 1, 2014