Orthogonal Derivations on Semirings

N. Suganthameena

Department of Mathematics
Kamaraj College of Engineering and Technology
Virudhunagar - 626001, India

M. Chandramouleeswaran

Department of Mathematics
Saiva Bhanu Kshatriya College
Aruppukottai - 626101, India

Copyright © 2014 N. Suganthameena and M. Chandramouleeswaran. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we introduce the concept of orthogonal derivations on Semirings and prove some results on semiprime semirings

Mathematics Subject Classification: 16Y30

Keywords: Semirings, Semiprime semiring, Derivations, Orthogonal derivations

1 Introduction

A semiring is an algebraic system $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups with two distributive laws. It has been formally introduced by H.S.Vandiver in 1934. A natural example of a semiring is the set of all natural numbers under usual addition and multiplication. The notion of derivation is quite old and is useful in the study of integration of analysis, Algebraic geometry and algebra.

The study of derivation in rings though initiated longback, got interested only after Posner who established two very striking results on derivation in prime rings in [4]. In [3] Jonathan Golan introduced the notion of derivations in semirings but nothing has been discussed in detail. This motivated Chandramouleeswaran and Triveni to study the notion of derivation on semirings in [1]. In [4] they discussed the notion of α derivation on semirings. In this paper we introduce the notion of orthogonal derivation on semirings and prove some simple but elegant results.

2 Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1 *A Semiring is a nonempty set S on which operations of addition and multiplication have been defined such that*

1. $(S, +)$ is a semigroup
2. $(S, \cdot)$ is a semigroup
3. Multiplication distributes over addition from either side

Definition 2.2 *Let $(S, +, \cdot)$ be a semiring. An element a in S is called additively left cancellative if $a + b = a + c \implies b = c \forall b, c \in S$. It is said to be right cancellative if $b + a = c + a \implies b = c$. It is said to be additively cancellative if it is both left and right cancellative. The Semiring is said to be additively cancellative if all the elements in S are additively cancellative*

Definition 2.3 *A Semiring S is said to be Prime if $aSb = 0 \implies a=0$ or $b=0$, $a, b \in S$*

Definition 2.4 *A Semiring S is said to be Semiprime if $aSa = 0 \implies a=0$, $a \in S$*

Definition 2.5 *A Semiring S is said to be 2 torsion free if $2a = 0 \implies a = 0$, $a \in S$*

Definition 2.6 *An additive mapping $d : S \rightarrow S$ is called a derivation if $(xy) = d(x)y + xd(y) \forall x, y \in S$*

Notation Throughout this paper we assume S to be the semiring with additive identity 0 and addition is commutative.

3 Orthogonal derivation on semirings

We begin this section with the definition of

Definition 3.1 Let S be a semiring. The derivations d and g of S into S is said to be orthogonal if $d(x)Sg(y) = g(x)Sd(y) \forall x, y \in S$

Example 3.2 Let $S = \{0, a, b, c\}$. Define the operations $+$ and $\cdot$ on S as follows

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

Then $(S, +, \cdot)$ is a semiring.

Define $d : S \rightarrow S$ such that $d(0) = 0$ and $d(x) = c$ when $x = a, b, c$

$g : S \rightarrow S$ such that $g(0) = 0$, $g(a) = b$ and $g(x) = c$ when $x = b, c$

clearly d and g are derivations on the semiring S .

Let $S_1 = S \oplus S$. Define $d_1 : S_1 \rightarrow S_1$ by $d_1(x, y) = (d(x), 0)$

$g_1 : S_1 \rightarrow S_1$ by $g_1(x, y) = (0, g(y))$, then d_1 and g_1 are orthogonal derivations.

Lemma 3.3 Let S be a 2 torsion free semiprime semiring, a and b the elements of S , then the following are equivalent.

(i) $asb = 0$

(ii) $bsa = 0$

(iii) $asb + bsa = 0$

If one of these conditions are fulfilled then $ab = ba = 0$

Proof Suppose $asb = 0$

Premultiplying by bs and post-multiplying by sa we have $(bsa)s(bsa) = 0$

Since S is semiprime we have $bsa = 0$

(ii) Suppose $bsa = 0$ Premultiplying by as and post-multiplying by sb we have $(asb)s(asb) = 0$

Since S is semiprime we have $asb = 0$

Then $asb + bsa = 0$

(iii) Suppose $asb + bsa = 0$

Premultiplying by bs we have $bs(asb) + bs(bsa) = 0$.

Again premultiplying by as we have

$$(asb)s(asb) + (asb)s(bsa) = 0 \quad (3.1)$$

Post-multiplying by sa we have, $(asb)sa + (bsa)sa = 0$.

Again postmultiplying by sb we have,

$$(asb)s(asb) + (bsa)s(asb) = 0 \quad (3.2)$$

Adding equations 3.1 and 3.2, and using (iii), we get $2((asb)s(asb)) = 0$

Since S is 2 torsion free and since S is semiprime, we have $asb = 0 \forall s \in S$.

Let $asb = 0$. Premultiplying by b and postmultiplying by a , we have

$$(ba)s(ba) = 0.$$

Since S is semiprime $ba = 0$.

Similarly from $bsa = 0$ we can prove that $ab = 0$.

Lemma 3.4 *Let S be a 2-torsion free semiprime semiring. Suppose that additive mappings f and g of a semiring S into S satisfying*

$$f(x)sg(x) = 0, \forall s \in S \text{ then } f(x)sg(y) = 0 \forall x, y \in S.$$

Proof Suppose $0 = f(x)sg(x)$ Replacing x by $x + y$ we have

$$\begin{aligned} 0 &= f(x+y)sg(x+y) \\ &= f(x)sg(x) + f(x)sg(y) + f(y)sg(x) + f(y)sg(y) \\ &= f(x)sg(y) + f(y)sg(x) \\ &= (f(x)sg(y))s_1(f(x)sg(y)) + f(x)s(g(y)s_1f(y))sg(x) \end{aligned}$$

By Lemma 3.3, $(f(x)sg(y))s_1(f(x)sg(y)) = 0$

Since S is semiprime, $f(x)sg(y) = 0 \forall x, y, s \in S$.

Lemma 3.5 *Let S be a 2-torsion free semiprime Semiring and let d and g be derivations of S into S . Derivations d and g are orthogonal iff $d(x)g(y) + g(x)d(y) = 0 \forall x, y \in S$.*

Proof Suppose $d(x)g(y) + g(x)d(y) = 0 \forall x, y \in S$. Replacing y by yx we get

$$\begin{aligned} 0 &= d(x)g(yx) + g(x)d(yx) \\ &= d(x)(g(y)x + yg(x)) + g(x)(d(y)x + yd(x)) \\ &= d(x)g(y)x + d(x)yg(x) + g(x)d(y)x + g(x)yd(x) \\ &= (d(x)g(y) + g(x)d(y))x + d(x)yg(x) + g(x)yd(x) \end{aligned}$$

By assumption $d(x)yg(x) + g(x)yd(x) = 0$

By Lemma 3.3, $d(x)yg(x) = g(x)yd(x) = 0 \forall x, y \in S$.

By lemma 3.4, $d(x)yg(z) = g(x)yd(z) = 0 \forall x, y, z \in S$ thus proving that d and g are orthogonal.

Conversely assume that d and g are orthogonal.

$$d(x)sg(y) = g(x)sd(y) = 0$$

By lemma 3.3, $d(x)g(y) = g(x)d(y) = 0$.

$$d(x)g(y) + g(x)d(y) = 0 \forall x, y \in S.$$

Lemma 3.6 *Let S be a 2-torsion free semiprime semiring. Suppose d and g are derivations of S into S . Then d and g are orthogonal iff $dg = 0$.*

Proof Suppose $dg = 0$.

$$\begin{aligned} 0 &= dg(xy) = d(g(x)y + xg(y)) \\ &= d(g(x))y + g(x)d(y) + d(x)g(y) + xd(g(y)) \\ &= g(x)d(y) + d(x)g(y) \end{aligned}$$

By lemma 3.5 d and g are orthogonal.

Conversely, since d and g are orthogonal $d(x)yg(z) = 0 \ \forall x, y, z \in S$.

$$\begin{aligned} d(d(x)yg(z)) &= 0 \\ &= d(d(x))yg(z) + d(x)d(y)g(z) + d(x)y d(g(z)) \\ &= d(x)y d(g(z)) \end{aligned}$$

Replacing x by $g(z)$ $d(g(z))y d(g(z)) = 0$.

Since S is semiprime, $d(g(z)) = 0 \ \forall z \in S$.

Hence $dg = 0$.

Lemma 3.7 *Let S be a 2-torsion free semiprime Semiring. Suppose d and g are derivations of S into S . Then d and g are orthogonal iff $dg + gd = 0$.*

Proof Suppose $dg + gd = 0$.

$$\begin{aligned} 0 &= (dg + gd)(xy) = dg(xy) + gd(xy) \\ &= d(g(x)y + xg(y)) + g(d(x)y + xd(y)) \\ &= dg(x)y + g(x)d(y) + d(x)g(y) + xdg(y) + gd(x)y + d(x)g(y) + g(x)d(y) + xg(d(y)) \\ &= (dg + gd(x)y + 2(d(x)g(y) + g(x)d(y))) + x(dg + gd)(y) \\ &= 2(d(x)g(y) + g(x)d(y)) \end{aligned}$$

Since S is 2-torsion free $d(x)g(y) + g(x)d(y) = 0$.

By lemma 3.5 d and g are orthogonal.

Conversely let d and g be orthogonal derivations.

By lemma 3.6 $dg = 0 = gd$. Then $dg + gd = 0$.

Lemma 3.8 *Let S be a 2-torsionfree semiprime semiring. Suppose d and g are derivations of S into S . Then d and g are orthogonal iff dg is a derivation.*

Proof Assume that dg is a derivation. Then $dg(xy) = dg(x)y + xdg(y)$

$$\begin{aligned} dg(xy) &= d(g(xy)) \\ &= d(g(x)y + xg(y)) \\ &= d(g(x))y + g(x)d(y) + d(x)g(y) + xd(g(y)) \end{aligned}$$

Comparing $g(x)d(y) + d(x)g(y) = 0$.

By lemma 3.5 d and g are orthogonal.

Conversely if d and g are orthogonal, by lemma 3.6 $dg = 0$.

Thus dg is a derivation.

Lemma 3.9 *Let S be a 2-torsion free semiprime semiring. If d is a derivation of S in to S , such that d^2 is a derivation then $d = 0$.*

Proof Since d^2 is a derivation $d^2(xy) = d^2(x)y + xd^2(y)$

$$\begin{aligned} d^2(xy) &= d(d(xy)) = d(d(x)y + xd(y)) \\ &= d^2(x)y + d(x)d(y) + d(x)d(y) + xd^2(y) \\ &= d^2(x)y + 2d(x)d(y) + xd^2(y) \end{aligned}$$

Comparing $2d(x)d(y) = 0$. Since S is 2-torsion free, $d(x)d(y) = 0 \forall x, y \in S$.

Replacing x by xs , we get

$$\begin{aligned} 0 &= d(xs)d(y) \\ &= (d(x)s + xd(s))d(y) \\ &= d(x)sd(y) + xd(s)d(y) \\ &= d(x)sd(y) \end{aligned}$$

$$\begin{aligned} \text{Replacing } y \text{ by } x + y, \text{ we get} &= d(x)sd(x + y) \\ &= d(x)s(d(x) + d(y)) \\ &= d(x)sd(x) + d(x)sd(y) = d(x)sd(x) \end{aligned}$$

Since S is semiprime $d(x) = 0 \forall x \in S$.

Lemma 3.10 *Let S be a semiprime semiring. Suppose that d and g are derivations of semiring S in to S . Then d and g are orthogonal iff there exists $a, b \in S$ such that $dg(x) = ax + xb \forall x \in S$.*

Proof Assume that $dg(x) = ax + xb \forall x \in S$.

Replacing x by xy , we get $dg(xy) = a(xy) + (xy)b$

Alternatively, $d(g(x)y + xg(y)) = a(xy) + (xy)b$

$$dg(x)y + g(x)d(y) + d(x)g(y) + xdg(y) = axy + xyb$$

$$xby + xay + d(x)g(y) + g(x)d(y) = 0 \quad \forall x, y \in S$$

Replacing y by yx

$$\begin{aligned} 0 &= xbyx + xayx + d(x)g(yx) + g(x)d(yx) \\ &= xbyx + xayx + d(x)g(y)x + d(x)yg(x) + g(x)d(y)x + g(x)yd(x) \\ &= (xby + xay + d(x)g(y) + g(x)d(y))x + d(x)yg(x) + g(x)yd(x) \\ &= d(x)yg(x) + g(x)yd(x) \end{aligned}$$

By lemmas 3.3 and 3.4 $d(x)yg(z) = 0 = g(x)yd(z)$.

This proves that d and g are orthogonal.

Conversely, if d and g are orthogonal by lemma 3.6 $dg = 0$

Then we can choose $a = b = 0$ so that $dg(x) = ax + xb$

References

- [1] **Chandramouleeswarn M. and Thiruveni V.** : On derivations of Semirings, *Advances in algebra* **vol 3** (2010), 123-131.
- [2] **Chandramouleeswarn M. and Thiruveni V.** : A note on α derivations in semirings, *International Journal of Pure and applied Science and Technology* (2011), 71-77.
- [3] **Jonathan S Golan**:Semirings and their applications,*Kluwar Academic press* ., (1969).
- [4] **Posner E.C.**, Derivation in Prime rings , American math soc Volume 8 (1957), 1093-1100. <http://dx.doi.org/10.1090/s0002-9939-1957-0095863-0>

Received: October 19, 2014; Published: November 19, 2014