Sums of Primes

An Asymptotic Expansion

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

Copyright © 2014 Rafael Jakimczuk. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this note we obtain an asymptotic expansion for $\sum_{p \leq x} p^\alpha$, where p denotes a positive prime and $\alpha > 0$.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Primes, sum of primes, asymptotic formulas

1 Introduction and Preliminary Notes

In this note we obtain an asymptotic expansion for $\sum_{p \leq x} p^\alpha$, where p denotes a positive prime and $\alpha > 0$.

We need the following well-known theorem, see [1, chapter XXII].

Theorem 1.1 *Let c_n ($n \geq 1$) a sequence of real numbers. Let us consider the function*

$$A(x) = \sum_{n \leq x} c_n$$

Suppose that $c_n = 0$ for $n < n_1$ and suppose that $f(x)$ has a continuous derivative $f'(x)$ on the interval $[n_1, \infty]$, then the following formula holds

$$\sum_{n \leq x} c_n f(n) = A(x)f(x) - \int_{n_1}^x A(t)f'(t) dt$$

Let $\pi(x)$ be the prime counting function. We also need the following well-known theorem, see [4, chapter 7].

Theorem 1.2 *Let m an arbitrary but fixed positive integer. The following formula holds*

$$\pi(x) = \sum_{k=1}^m \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^m x}\right)$$

In this note p denotes a positive prime.

2 Main Results

Theorem 2.1 *Let m be an arbitrary but fixed positive integer and α an arbitrary but fixed positive real number. The following asymptotic expansion holds*

$$\sum_{p \leq x} p^\alpha = \sum_{k=1}^m \frac{(k-1)!x^{\alpha+1}}{(\alpha+1)^k \log^k x} + o\left(\frac{x^{\alpha+1}}{\log^m x}\right) \quad (1)$$

Proof. If we put $c_n = 1$ if n is prime and $c_n = 0$ if n is not prime then we have (theorem 1.2)

$$\sum_{n \leq x} c_n = \sum_{p \leq x} 1 = \pi(x) = \sum_{k=1}^m \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^m x}\right) \quad (2)$$

If $f(x) = x^\alpha$ then theorem 1.1 gives

$$\begin{aligned} \sum_{p \leq x} p^\alpha &= \sum_{k=1}^m \frac{(k-1)!x^{\alpha+1}}{\log^k x} + o\left(\frac{x^{\alpha+1}}{\log^m x}\right) - \alpha \left(\sum_{k=1}^m \int_2^x \frac{(k-1)!t^\alpha}{\log^k t} dt \right) \\ &+ \int_2^x o\left(\frac{t^\alpha}{\log^m t}\right) dt \end{aligned} \quad (3)$$

We have

$$\int_2^x o\left(\frac{t^\alpha}{\log^m t}\right) dt = o\left(\int_2^x \frac{t^\alpha}{\log^m t} dt\right) = o\left(\frac{x^{\alpha+1}}{\log^m x}\right) \quad (4)$$

since (L'Hospital's rule)

$$\lim_{x \rightarrow \infty} \frac{\int_2^x \frac{t^\alpha}{\log^m t} dt}{\frac{x^{\alpha+1}}{(\alpha+1)\log^m x}} = 1$$

On the other hand, we have the formula ($m \geq 1$).

$$\begin{aligned} \sum_{k=1}^m \int_2^x \frac{(k-1)!t^\alpha}{\log^k t} dt &= \sum_{k=1}^m \left(\sum_{i=1}^k \frac{(k-1)!}{(\alpha+1)^i} \right) \frac{x^{\alpha+1}}{\log^k x} \\ &+ O(1) + \left(\sum_{i=1}^m \frac{m!}{(\alpha+1)^i} \right) \int_2^x \frac{t^\alpha}{\log^{m+1} t} dt \end{aligned} \quad (5)$$

This formula can be proved using mathematical induction on m and the formula (use integration by parts)

$$\int_2^x \frac{t^\alpha}{\log^r t} dt = \frac{x^{\alpha+1}}{(\alpha+1)\log^r x} + O(1) + \frac{r}{\alpha+1} \int_2^x \frac{t^\alpha}{\log^{r+1} t} dt \quad (r \geq 1)$$

Now (L'Hospital's rule)

$$\lim_{x \rightarrow \infty} \frac{\int_2^x \frac{t^\alpha}{\log^{m+1} t} dt}{\frac{x^{\alpha+1}}{\log^m x}} = 0 \quad (6)$$

Therefore (5) and (6) give

$$\sum_{k=1}^m \int_2^x \frac{(k-1)! t^\alpha}{\log^k t} dt = \sum_{k=1}^m \left(\sum_{i=1}^k \frac{(k-1)!}{(\alpha+1)^i} \right) \frac{x^{\alpha+1}}{\log^k x} + o\left(\frac{x^{\alpha+1}}{\log^m x}\right) \quad (7)$$

Substituting (4) and (7) into (3) we obtain (1), since

$$(k-1)! - \alpha(k-1)! \sum_{i=1}^k \frac{1}{(\alpha+1)^i} = \frac{(k-1)!}{(\alpha+1)^k}$$

We have used the well-known formula

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

The theorem is proved.

Example 2.2 If $m = 1$ then equation (1) becomes

$$\sum_{p \leq x} p^\alpha = \frac{x^{\alpha+1}}{(\alpha+1)\log x} + o\left(\frac{x^{\alpha+1}}{\log x}\right) \quad (8)$$

If $m = 2$ then equation (1) becomes

$$\sum_{p \leq x} p^\alpha = \frac{x^{\alpha+1}}{(\alpha+1)\log x} + \frac{x^{\alpha+1}}{(\alpha+1)^2 \log^2 x} + o\left(\frac{x^{\alpha+1}}{\log^2 x}\right) \quad (9)$$

3 An Application

In this section p_n denotes the n -th prime number.

Equation (8) is well-known (see, for example, [2]). This equation can be written in the form

$$\sum_{p \leq x} p^\alpha \sim \frac{x^{\alpha+1}}{(\alpha+1)\log x} \quad (10)$$

An immediate consequence of (10) and the prime number theorem $p_n \sim n \log n$ is the equation

$$\sum_{i=1}^n p_i^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n \quad (11)$$

since the prime number theorem implies $\log p_n \sim \log n$.

Equation (11) is well known (see, for example, either [2] or [3]). In this section we prove the more precise theorem.

Theorem 3.1 *Let α be a positive real number. The following asymptotic formula holds*

$$\begin{aligned} \sum_{i=1}^n p_i^\alpha &= \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n + \frac{\alpha}{\alpha+1} n^{\alpha+1} \log^\alpha n \frac{\log \log n}{\log n} \\ &\quad + o\left(n^{\alpha+1} \log^\alpha n \frac{\log \log n}{\log n}\right) \end{aligned} \quad (12)$$

Proof. Equation (9) can be written in the form

$$\sum_{p \leq x} p^\alpha = \frac{x^{\alpha+1}}{(\alpha+1) \log x} + f(x) \frac{x^{\alpha+1}}{(\alpha+1)^2 \log^2 x} \quad (13)$$

where $\lim_{x \rightarrow \infty} f(x) = 1$.

Equation (13) gives

$$\sum_{i=1}^n p_i^\alpha = \sum_{p \leq p_n} p^\alpha = \frac{p_n^{\alpha+1}}{(\alpha+1) \log p_n} + f(p_n) \frac{p_n^{\alpha+1}}{(\alpha+1)^2 \log^2 p_n} \quad (14)$$

Now, we have $p_n \sim n \log n$ and $\log p_n \sim \log n$. Therefore

$$f(p_n) \frac{p_n^{\alpha+1}}{(\alpha+1)^2 \log^2 p_n} \sim \frac{n^{\alpha+1}}{(\alpha+1)^2} \log^{\alpha-1} n \quad (15)$$

We have (see, for example, [3]) the following formula

$$p_n = n \log n + n \log \log n + o(n \log \log n) \quad (16)$$

Hence

$$\begin{aligned} p_n^{\alpha+1} &= (n \log n + n \log \log n + o(n \log \log n))^{\alpha+1} \\ &= n^{\alpha+1} \log^{\alpha+1} n \left(1 + \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)\right)^{\alpha+1} \\ &= n^{\alpha+1} \log^{\alpha+1} n \left(1 + (\alpha+1) \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)\right) \end{aligned} \quad (17)$$

where we have used the formula $(1+x)^\beta = 1 + \beta x + o(x)$ ($x \rightarrow 0$).

Note that the prime number theorem $p_n \sim n \log n$ implies

$$\log p_n = \log n + \log \log n + o(1) = \log n + \log \log n + o(\log \log n)$$

Therefore we have

$$\begin{aligned} \frac{1}{\log p_n} &= \frac{1}{\log n + \log \log n + o(\log \log n)} = \frac{1}{\log n} \frac{1}{1 + \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)} \\ &= \frac{1}{\log n} \left(1 - \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)\right) \end{aligned} \quad (18)$$

where we have used the formula $\frac{1}{1+x} = 1 - x + o(x)$ ($x \rightarrow 0$).

Equations (17) and (18) give

$$\begin{aligned} \frac{1}{\alpha+1} \frac{p_n^{\alpha+1}}{\log p_n} &= \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} \left(1 + (\alpha+1) \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)\right) \\ &\quad \left(1 - \frac{\log \log n}{\log n} + o\left(\frac{\log \log n}{\log n}\right)\right) = \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} \\ &\quad + \frac{\alpha}{\alpha+1} n^{\alpha+1} \log^{\alpha-1} n \log \log n + o\left(n^{\alpha+1} \log^{\alpha-1} n \log \log n\right) \end{aligned} \quad (19)$$

Substituting (19) and (15) into (14) we obtain (12). The theorem is proved.

Remark 3.2 If α is a positive integer more precise formulae are obtained in [3]

ACKNOWLEDGEMENTS. The author is very grateful to Universidad Nacional de Luján.

References

- [1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Oxford, 1960.
- [2] R. Jakimczuk, A note on sums of powers which have a fixed number of prime factors, *Journal of Inequalities in Pure and Applied Mathematic*, **6** (2005), article 31.
- [3] R. Jakimczuk, Desigualdades y fórmulas asintóticas para sumas de potencias de primos, *Boletín de la Sociedad Matemática Mexicana*, **11** (2005), 5 - 10.
- [4] W. J. LeVeque, *Topics in Number Theory*, Volume 2, Addison-Wesley, 1958.

Received: November 18, 2014; Published: December 30, 2014