The Euler's Function and the Distribution of Certain Numbers

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Abstract

In this note we show that the arithmetical function $\varphi(n)/n^2$ is related with the distribution of certain positive integers, where $\varphi(n)$ is the Euler's function.

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1 Introduction and Preliminary Results

In this note $\lfloor x \rfloor$ denotes the integer-part function. Note that

$$0 \leq x - \lfloor x \rfloor < 1. \quad (1)$$

We shall need the following theorem, it is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

Theorem 1.1 *Let S be a set of N distinct elements, and let $S_1, \dots, S_r$ be arbitrary subsets of S containing $N_1, \dots, N_r$ elements, respectively. For $1 \leq i < j < \dots < l \leq r$, let $S_{ij\dots l}$ be the intersection of $S_i, S_j, \dots, S_l$ and let*

$N_{ij\dots l}$ be the number of elements of $S_{ij\dots l}$. Then the number K of elements of S not in any of $S_1, \dots, S_r$ is

$$K = N - \sum_{1 \leq i \leq r} N_i + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \dots + (-1)^r N_{12\dots r}. \quad (2)$$

Proof. See, for example, either [2, page 233] or [3, page 84].

If $n \geq 1$ the Euler's function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n . The following formula is well-known

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad (3)$$

where, the product run on the different prime divisors p of n .

In this note we show that the arithmetical function $\varphi(n)/n^2$ is relationed with the distribution of certain positive integers.

2 Main Result

Let us consider the numbers of the form $p_1^{h_1} \dots p_k^{h_k} C$, where $p_1, \dots, p_k$ are $k \geq 1$ different primes fixed, $h_1, \dots, h_k$ are $k \geq 1$ positive integers fixed, the positive integer C is variable and C and $p_1^{h_1} \dots p_k^{h_k}$ are relatively prime. The set of these numbers is denoted $S_{p_1^{h_1} \dots p_k^{h_k}}$ and the number of these numbers not exceeding x is denoted $A_{p_1^{h_1} \dots p_k^{h_k}}(x)$.

Theorem 2.1 *The numbers in the set $S_{p_1^{h_1} \dots p_k^{h_k}}$ have positive density $D_{p_1^{h_1} \dots p_k^{h_k}}$, where*

$$D_{p_1^{h_1} \dots p_k^{h_k}} = \frac{(p_1 - 1) \dots (p_k - 1)}{p_1^{h_1+1} \dots p_k^{h_k+1}} = \frac{\varphi(p_1^{h_1} \dots p_k^{h_k})}{(p_1^{h_1} \dots p_k^{h_k})^2} \quad (4)$$

Proof. Equations (2), (1) and (3) give

$$\begin{aligned} & A_{p_1^{h_1} \dots p_k^{h_k}}(x) \\ &= \left\lfloor \frac{x}{p_1^{h_1} \dots p_k^{h_k}} \right\rfloor - \left\lfloor \frac{x}{p_1^{h_1+1} \dots p_k^{h_k}} \right\rfloor - \dots - \left\lfloor \frac{x}{p_1^{h_1} \dots p_k^{h_k+1}} \right\rfloor + \dots \\ &= \frac{x}{p_1^{h_1} \dots p_k^{h_k}} - \frac{x}{p_1^{h_1+1} \dots p_k^{h_k}} - \dots - \frac{x}{p_1^{h_1} \dots p_k^{h_k+1}} + \dots + O(1) \\ &= \frac{1}{p_1^{h_1} \dots p_k^{h_k}} \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_k}\right) x + O(1) \end{aligned}$$

$$\begin{aligned}
&= \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} x + O(1) \\
&= \frac{\varphi(p_1^{h_1} \cdots p_k^{h_k})}{(p_1^{h_1} \cdots p_k^{h_k})^2} x + O(1)
\end{aligned} \tag{5}$$

Equation (5) gives

$$D_{p_1^{h_1} \cdots p_k^{h_k}} = \lim_{x \rightarrow \infty} \frac{A_{p_1^{h_1} \cdots p_k^{h_k}}(x)}{x} = \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} = \frac{\varphi(p_1^{h_1} \cdots p_k^{h_k})}{(p_1^{h_1} \cdots p_k^{h_k})^2}$$

That is, equation (4). The theorem is proved.

The set of the multiples of $p_1 \cdots p_k$ we denote $E_{p_1 \cdots p_k}$, the density of this set is clearly $\frac{1}{p_1 \cdots p_k}$. Note that the union of the infinite disjoint sets $S_{p_1^{h_1} \cdots p_k^{h_k}}$ is the set $E_{p_1 \cdots p_k}$. That is

$$\bigcup_{(h_1, \dots, h_k)} S_{p_1^{h_1} \cdots p_k^{h_k}} = E_{p_1 \cdots p_k}$$

where the symbol

$$\bigcup_{(h_1, \dots, h_k)}$$

denotes that the union run on all possible vectors $(h_1, \dots, h_k)$.

In the next theorem we prove that the sum of the infinity densities of the infinite disjoint sets $S_{p_1^{h_1} \cdots p_k^{h_k}}$ is the density of the union of these sets, namely, the density of the set $E_{p_1 \cdots p_k}$.

Theorem 2.2 *The following formula holds*

$$\sum_{(h_1, \dots, h_k)} D_{p_1^{h_1} \cdots p_k^{h_k}} = \frac{1}{p_1 \cdots p_k} \tag{6}$$

Proof. We have (see equation (4))

$$\begin{aligned}
\sum_{(h_1, \dots, h_k)} D_{p_1^{h_1} \cdots p_k^{h_k}} &= \sum_{(h_1, \dots, h_k)} \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} \\
&= (p_1 - 1) \cdots (p_k - 1) \left(\frac{1}{p_1^2} + \frac{1}{p_1^3} + \cdots \right) \cdots \left(\frac{1}{p_k^2} + \frac{1}{p_k^3} + \cdots \right) \\
&= (p_1 - 1) \cdots (p_k - 1) \frac{1}{p_1^2} \frac{1}{1 - \frac{1}{p_1}} \cdots \frac{1}{p_k^2} \frac{1}{1 - \frac{1}{p_k}} = \frac{1}{p_1 \cdots p_k}
\end{aligned}$$

where we have use the well-known series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \quad |x| < 1$$

The theorem is proved.

Note that if we put $n = p_1^{h_1} \cdots p_k^{h_k}$, then the numbers of the form $p_1^{h_1} \cdots p_k^{h_k} C = nC$ have density (see equation (4))

$$D_n = \frac{\varphi(n)}{n^2} \quad (n = 1, 2, 3, \dots) \quad (7)$$

Besides, we have the following theorem.

Theorem 2.3 *The following formula holds*

$$\sum_{n \leq x} D_n = \sum_{n \leq x} \frac{\varphi(n)}{n^2} = \frac{6}{\pi^2} \log x + \frac{6}{\pi^2} \gamma - B + O\left(\frac{\log x}{x}\right)$$

where γ is the Euler's constant,

$$B = \sum_{m=1}^{\infty} \frac{\mu(m) \log m}{m^2}$$

and $\mu(m)$ is the Mobius's function.

Proof. See [1, chapter 3].

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