Finite Difference Method of Fractional Parabolic Partial Differential Equations with Variable Coefficients

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Abstract

In this study, we consider the fractional parabolic partial differential equations that variable coefficients with the Caputo fractional derivative. We constructed a difference scheme based on Crank- Nicholson method. And stability of this difference scheme is proved conditionally.

Mathematics Subject Classification: 65N06, 65N12, 65M12

Keywords: Difference scheme, Stability, Caputo fractional derivative

1 Introduction

In this study, we propose a Crank-Nicholson type of difference scheme and we see that under some conditions the difference scheme is stable for the following time fractional partial differential problem that variable coefficients;

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x) \frac{\partial^2 u(x,t)}{\partial x^2} - b(x) \frac{\partial u(x,t)}{\partial x}, \\ -c(x) u(x,t) + f(x,t), (0 < x < 1, 0 < t < 1), \\ u(x,0) = r(x), 0 \leq x \leq 1, \\ u(0,t) = 0, u(1,t) = 0, 0 \leq t \leq 1. \end{cases} \quad (1)$$

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(x, \tau)}{(t-\tau)^\alpha} d\tau, 0 < \alpha < 1, \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function.

We can obtain some kind of subproblems from this problem which general type. One can see that if $b(x) = 0$ and $c(x) = 0$ then the problem transforms to time fractional diffusion problem that variable coefficient, if we choose $b(x) = 0$ and $c(x) > 0$ then the problem will be a time-fractional cable problem that variable coefficient when opposite chosen which means $c(x) = 0$ and $b(x) > 0$ the problem is called fractional advection dispersion problem that variable coefficient.

2 Discretization of the Problem

For some positive integers M and N , the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0, 1]$ are the numbers $x_j = jh$, $j = 0, 1, 2, \dots, M$, and the grid points in the time interval $[0, 1]$ are labeled $t_k = k\tau$, $k = 0, 1, 2, \dots, N$. The values of the functions u and f at the grid points are denoted $u_j^k = u(x_j, t_k)$ and $f_j^k = f(x_j, t_k)$, respectively. Let $u(x, t)$, $u_t(x, t)$ and $u_{tt}(x, t)$ are continuous on $[0, 1]$.

A discrete approximation to the fractional derivative $\frac{\partial^\alpha u(x, t)}{\partial t^\alpha}$ at $(x_j, t_{k+\frac{1}{2}})$ can be obtained by the following approximation[7]:

$$\begin{aligned} \frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} &= \left[w_1 u^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u^m - w_k u^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] \\ &+ O(\tau^{2-\alpha}). \end{aligned} \quad (3)$$

Where $\sigma = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau^\alpha}$ and $w_j = \sigma ((j+1/2)^{1-\alpha} - (j-1/2)^{1-\alpha})$. In addition for $k = 0$ there is no these terms $w_1 u_k$ and $w_k u_0$. On the other hand, we have also approximations for second and first derivative at $(x_j, t_{k+\frac{1}{2}})$.

$$\frac{\partial^2 u(x_j, t_{k+\frac{1}{2}})}{\partial x^2} = \frac{1}{2} \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} \right] + O(h^2). \quad (4)$$

$$\frac{\partial u(x_j, t_{k+\frac{1}{2}})}{\partial x} = \frac{1}{2} \left[\frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2h} + \frac{u_{j+1}^k - u_{j-1}^k}{2h} \right] \quad (5)$$

Using these approximations (3), (4) and (5) into (1), we obtain the following difference scheme for (1) which is accurate of order $O(\tau^{2-\alpha} + h^2)$;

$$\left\{ \begin{array}{l} \left[w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] \\ - a(x_j) \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{2h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{2h^2} \right] + b(x_j) \left(\frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{4h} + \frac{u_{j+1}^k - u_{j-1}^k}{4h} \right) \\ + c(x_j) \frac{u_j^{k+1} + u_j^k}{2} = f(x_j, t_k + \frac{\tau}{2}), \\ 0 \leq k \leq N-1, \quad 1 \leq j \leq M-1, \\ \\ u_j^0 = r(x_j), \quad 0 \leq j \leq M, \\ u_0^k = 0, \quad u_M^k = 0, \quad 0 \leq k \leq N. \end{array} \right.$$

$$\left\{ \begin{array}{l} \left(-\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right) u_{j+1}^k + \left(-\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right) u_{j+1}^{k+1} + \left(-\frac{\sigma}{2^{1-\alpha}} + \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2} \right) u_j^k \\ + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2} \right) u_j^{k+1} + w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 \\ + \left(-\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right) u_{j-1}^k + \left(-\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right) u_{j-1}^{k+1} \\ = f(x_j, t_k + \frac{\tau}{2}), \quad 0 \leq k \leq N-1, \quad 1 \leq j \leq M-1, \\ u_j^0 = r(x_j), \quad 0 \leq j \leq M, \\ u_0^k = 0, \quad u_M^k = 0, \quad 0 \leq k \leq N. \end{array} \right. \quad (6)$$

3 Matrix Stability of the Difference Scheme

And we constructed this difference scheme from (6)

$$\left\{ \begin{array}{l} \left(-\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right) u_{j-1}^{k+1} + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2} \right) u_j^{k+1} + \left(-\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right) u_{j+1}^{k+1} \\ + \left(-\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right) u_{j-1}^k + \left(-\frac{\sigma}{2^{1-\alpha}} + \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2} \right) u_j^k + \left(-\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right) u_{j+1}^k \\ + w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 \\ = f(x_j, t_k + \frac{\tau}{2}), \quad 0 \leq k \leq N-1, \quad 1 \leq j \leq M-1, \\ u_j^0 = r(x_j), \quad 0 \leq j \leq M, \\ u_0^k = 0, \quad u_M^k = 0, \quad 0 \leq k \leq N. \end{array} \right. \quad (7)$$

The difference scheme above (7) can be written in matrix form,

$$\begin{cases} AU^1 = BU^0 + \varphi^0, & k = 0 \\ AU^{k+1} = BU^k - w_1 U^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) U^m \\ + w_k U^0 + \varphi^k, & 1 \leq k \leq N-1 \\ U_j^0 = r(x_j), & 0 \leq j \leq M, \\ U_0^k = 0, & U_M^k = 0, \quad 0 \leq k \leq N. \end{cases} \quad (8)$$

where $\varphi^k = [\varphi_1^k, \varphi_2^k, \varphi_3^k, \dots, \varphi_{M-1}^k]^T$, $\varphi_j^0 = r(x_j)$, $\varphi_j^k = f(x_j, t_{k+1/2})$, $1 \leq k \leq N$, $1 \leq j \leq M$, and $U^k = [U_1^k, U_2^k, U_3^k, \dots, U_{M-1}^k]^T$. Here, A and B are 3-diagonal $(M-1) \times (M-1)$ matrices of the form :

$$A = \begin{bmatrix} -\frac{\sigma}{2^{1-\alpha}} + p & -\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} & & & \\ -\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} & \frac{\sigma}{2^{1-\alpha}} + p & -\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \\ & & & -\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} & -\frac{\sigma}{2^{1-\alpha}} + p \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\sigma}{2^{1-\alpha}} - p & \frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} & & & \\ \frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} & \frac{\sigma}{2^{1-\alpha}} - p & \frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} & & \\ & \frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \\ & & & \frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} & \frac{\sigma}{2^{1-\alpha}} - p \end{bmatrix}$$

where $p = \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2}$

We denote $\|A\| = \|A\|_\infty = \max_{1 \leq j \leq M-1} \left\{ \sum_{m=1}^{M-1} |a_{jm}| \right\}$, where $A = [a_{jm}]_{(M-1) \times (M-1)}$.

Lemma3.1 If $\frac{\sigma}{2^{1-\alpha}} - \frac{a(x_j)}{h^2} - \frac{c(x_j)}{2} > 0$, $\frac{a(x_j)}{b(x_j)} > \frac{h}{2}$, $a(x_j) > 0$, $b(x_j) > 0$ and $c(x_j) > 0$, then $\|A^{-1}B\| \leq 1$.

Proof. Therefore,

$$\begin{aligned} \|A^{-1}B\| &\leq \|A^{-1}\| \|B\| \leq \frac{1}{\min_{1 \leq j \leq M-1} \left\{ |a_{jj}| - \sum_{m \neq j, m=1}^{M-1} |a_{jm}| \right\}} \|B\| \\ &\leq \frac{\left| \frac{\sigma}{2^{1-\alpha}} - \frac{a(x_j)}{h^2} - \frac{c(x_j)}{2} \right| + \left\{ \left| \frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right| + \left| \frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right| \right\}}{\left| \frac{\sigma}{2^{1-\alpha}} + \frac{a(x_j)}{h^2} + \frac{c(x_j)}{2} \right| - \left\{ \left| -\frac{a(x_j)}{2h^2} - \frac{b(x_j)}{4h} \right| + \left| -\frac{a(x_j)}{2h^2} + \frac{b(x_j)}{4h} \right| \right\}} \\ &= \frac{\frac{\sigma}{2^{1-\alpha}} - \frac{c(x_j)}{2}}{\frac{\sigma}{2^{1-\alpha}} + \frac{c(x_j)}{2}} \leq 1. \end{aligned}$$

Lemma3.2 If $\frac{\sigma}{2^{1-\alpha}} - \frac{a(x_j)}{h^2} - \frac{c(x_j)}{2} - w_1 > 0$, $\frac{a(x_j)}{b(x_j)} > \frac{h}{2}$, $a(x_j) > 0$, $b(x_j) > 0$ and $c(x_j) > 0$ then $\|A^{-1}(B - w_1 I)\| + \|A^{-1}w_1\| \leq 1$

Proof.

$$\begin{aligned}
\|A^{-1}(B - w_1 I)\| + \|w_1 A^{-1}\| &\leq \|A^{-1}\| \|B - w_1 I\| + \|w_1 A^{-1}\| \\
&\leq \frac{\frac{\sigma}{2^{1-\alpha}} - \frac{c(x_j)}{2} - w_1}{\frac{\sigma}{2^{1-\alpha}} + \frac{c(x_j)}{2}} + \frac{w_1}{\frac{\sigma}{2^{1-\alpha}} + \frac{c(x_j)}{2}} \\
&\leq 1
\end{aligned}$$

Theorem 3.1 The difference scheme (8) is stable.

Proof. To prove the conditional stability of (8), let U_j^k and V_j^k be the exact and approximate solution of (8) with initial value U_j^0 and V_j^0 respectively. We denote the corresponding error by $\varepsilon_j^k = U_j^k - V_j^k$ and $\varepsilon^k = [\varepsilon_1^k, \varepsilon_2^k, \dots, \varepsilon_{M-1}^k]^t$ where $(0 \leq j \leq M, 0 \leq k \leq N)$. Then ε^k satisfies if $k = 0$

$$A\varepsilon^1 = B\varepsilon^0$$

if $k > 0$

$$A\varepsilon^{k+1} = B\varepsilon^k - w_1 \varepsilon^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \varepsilon^m + w_k \varepsilon^0.$$

Let us prove $\|\varepsilon^k\| \leq \|\varepsilon^0\|$, $k = 0, 1, 2, \dots$ by induction. In fact, if $k = 0$

$$\varepsilon^1 = A^{-1}B\varepsilon^0$$

from that

$$\|\varepsilon^1\| = \|A^{-1}B\varepsilon^0\| \leq \|A^{-1}B\| \|\varepsilon^0\|.$$

Since $\|A^{-1}B\| \leq 1$, from the Lemma 1, we have $\|\varepsilon^1\| \leq \|\varepsilon^0\|$. If $k = 1$, then we have

$$\varepsilon^2 = A^{-1}(B - w_1 I)\varepsilon^1 + w_1 A^{-1}\varepsilon^0.$$

From the last equation, we obtain

$$\begin{aligned}
\|\varepsilon^2\| &= \|A^{-1}(B - w_1 I)\varepsilon^1 + w_1 A^{-1}\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^1\| + \|w_1 A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^0\| + \|w_1 A^{-1}\| \|\varepsilon^0\| \\
&\leq \{\|A^{-1}(B - w_1 I)\| + \|w_1 A^{-1}\|\} \|\varepsilon^0\|.
\end{aligned}$$

If the condition above is satisfied, then $\|\varepsilon^2\| \leq \|\varepsilon^0\|$ is obtained. Now, assume

$\|\varepsilon^s\| \leq \|\varepsilon^0\|$ for all $s \leq k$, we will prove it is also true for $s = k + 1$.

$$\begin{aligned}
\|\varepsilon^{k+1}\| &= \left\| A^{-1} \left(B\varepsilon^k - w_1\varepsilon^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1})\varepsilon^m + w_k\varepsilon^0 \right) \right\| \\
&\leq \|A^{-1}(B - w_1I)\| \|\varepsilon^k\| + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \|A^{-1}\| \|\varepsilon^m\| \\
&\quad + w_k \|A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1I)\| \|\varepsilon^0\| + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \|A^{-1}\| \|\varepsilon^0\| \\
&\quad + w_k \|A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1I)\| \|\varepsilon^0\| + \{(w_{k-1} - w_k) + \dots + (w_1 - w_2) + w_k\} \|A^{-1}\| \|\varepsilon^0\| \\
&\leq (\|A^{-1}(B - w_1I)\| + w_1 \|A^{-1}\|) \|\varepsilon^0\| \\
&\leq \|\varepsilon^0\|.
\end{aligned}$$

Therefore, under the condition $\frac{\sigma}{2^{1-\alpha}} - \frac{a(x_j)}{h^2} - \frac{c(x_j)}{2} - w_1 > 0$, $\frac{a(x_j)}{b(x_j)} > \frac{h}{2}$, $a(x_j) > 0$, $b(x_j) > 0$ and $c(x_j) > 0$, the stability inequality is obtained.

4 Convergence Of The Difference Scheme

Theorem4.1 The proposed scheme (8) is convergent and the following estimate holds:

$$\|e^k\| \leq Z(\alpha)(\tau^{2-\alpha} + h^2),$$

with the condition $\frac{\sigma}{2^{1-\alpha}} - \frac{a(x_j)}{h^2} - \frac{c(x_j)}{2} - w_1 > 0$, $\frac{a(x_j)}{b(x_j)} > \frac{h}{2}$, $a(x_j) > 0$, $b(x_j) > 0$ and $c(x_j) > 0$. Here $Z(\alpha)$ does not depend on τ and h .

Proof. Let $R = [c(\tau^{2-\alpha} + h^2), \dots, c(\tau^{2-\alpha} + h^2)]_{M-1}^T$ set $w_0^{-1} = 1$. Since $e_j^k = u_j^k - U_j^k$, notice that $e^0 = 0$. Firstly, we prove that $\|e^{k+1}\| \leq w_n^{-1} \|R\|$ for all n by induction. We have the following error equation when $k = 0$.

$$\begin{aligned}
e^1 &= A^{-1}Be^0 + A^{-1}R = A^{-1}R \\
\|e^1\| &= \|A^{-1}R\| \leq \|A^{-1}\| \|R\| \leq w_0^{-1} \|R\|.
\end{aligned}$$

Similarly, for $k = 2$ we have the error equation

$$e^2 = A^{-1}(B - w_1I)e^1 + w_1A^{-1}e^0 + A^{-1}R$$

If we take the norm of this equality, we obtain

$$\begin{aligned}
\|e^2\| &= \|A^{-1}(B - w_1I)\| \|e^1\| + \|A^{-1}R\| \\
&\leq [\|A^{-1}(B - w_1I)\| + \|A^{-1}w_1\|] .w_1^{-1} \|R\| \\
&\leq w_1^{-1} \|R\|.
\end{aligned}$$

Assume the inequality $\|e^s\| \leq w_{s-1}^{-1} \|R\|$ is true for all $s \leq k$. Now, we will prove that it is also true for $s = k + 1$. We have the following error equation

$$e^{k+1} = A^{-1} \left(B e^k - w_1 e^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) e^m + w_k e^0 + R \right)$$

$$\begin{aligned} \|e^{k+1}\| &\leq \|A^{-1} (B - w_1 I)\| \|e^k\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1} |w_{k-m} - w_{k-m+1}| \| \|e^m\| + \|A^{-1} R\| \\ &\leq \|A^{-1} (B - w_1 I)\| w_{k-1}^{-1} \|R\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1} |w_{k-m} - w_{k-m+1}| \| w_{m-1}^{-1} \|R\| + \|A^{-1} w_k\| w_k^{-1} \|R\| \\ &\leq \|A^{-1} (B - w_1 I)\| w_k^{-1} \|R\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1} |w_{k-m} - w_{k-m+1}| \| w_k^{-1} \|R\| + \|A^{-1} w_k\| w_k^{-1} \|R\| \\ &\leq \left(\|A^{-1} (B - w_1 I)\| + \sum_{m=1}^{k-1} \|A^{-1} |w_{k-m} - w_{k-m+1}| \| + \|A^{-1} w_k\| \right) w_k^{-1} \|R\| \\ &\leq (\|A^{-1} (B - w_1 I)\| + \|A^{-1} w_1\|) w_k^{-1} \|R\| \\ &\leq w_k^{-1} \|R\| \end{aligned}$$

Since,

$$\lim_{k \rightarrow \infty} \frac{k^{-\alpha}}{(k + \frac{1}{2})^{1-\alpha} - (k - \frac{1}{2})^{1-\alpha}} = \frac{1}{1-\alpha}$$

there exists constant $C > 0$ such that

$$\begin{aligned} \|e^{k+1}\| &\leq w_k^{-1} \|R\| = k^\alpha k^{-\alpha} w_k^{-1} \|R\| = \frac{(k^\alpha \tau^\alpha) k^{-\alpha} \Gamma(2-\alpha)}{(k + \frac{1}{2})^{1-\alpha} - (k - \frac{1}{2})^{1-\alpha}} \|R\| \\ &\leq C \frac{\Gamma(2-\alpha)}{1-\alpha} (\tau^{2-\alpha} + h^2) \\ &\leq Z(\alpha) (\tau^{2-\alpha} + h^2), \end{aligned}$$

where $Z(\alpha) = C \frac{\Gamma(2-\alpha)}{1-\alpha}$. So, corresponding difference scheme is convergent under the condition.

5 Numerical Example

Consider this problem,

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = x \frac{\partial^2 u(t,x)}{\partial x^2} - \sin(x) \frac{\partial u(t,x)}{\partial x} - x^2 u(t,x) \\ + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} (1-x) \sin(x) + t^2 [(1-x) \sin(x)(x + \cos(x) + x^2) + 2x \cos(x) - \sin^2(x)] , \\ (0 < x < 1, 0 < t < 1), \\ u(0, x) = 0, 0 \leq x \leq 1, \\ u(t, 0) = 0, u(t, 1) = 0, 0 \leq t \leq 1. \end{cases}$$

Exact solution of this problem is $u(t, x) = t^2(1-x) \sin(x)$. The errors for some

M and N are given in figure 1. The errors when solving this problem are listed in the table1 for various values of time and space nodes. The errors in the table 1 are calculated by the formula $\max_{\substack{0 \leq n \leq M \\ 0 \leq k \leq N}} |u(t_k, x_n) - U_n^k|$.

Figure 1: The errors for some values of M and N when $t = 1$.

Table 1: The errors for some values of M, N and α

N	M	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
		$Error(\alpha, \tau)$	Err. rate	$Error(\alpha, \tau)$	Err. rate	$Error(\alpha, \tau)$	Err. rate
8	16	0.00174746	-	0.00154891	-	0.0012995	-
16	16	0.00043279	4.03	0.00037417	4.1	0.0003855	3.37
32	16	0.00010715	4.03	0.00009049	4.1	0.0001205	3.19

6 Conclusion

In this work, $O(\tau^{2-\alpha} + h^2)$ order approximation for the Caputo fractional derivative based on the Crank-Nicholson difference scheme was successfully applied to solve the fractional parabolic partial differential equations with variable coefficients. It is proven that the time-fractional Crank-Nicholson difference scheme is conditionally stable. Numerical results are agreement with the theoretical results.

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