Equienergetic Net-regular Signed Graphs

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Abstract

The concept of energy is recently generalized to signed graphs i.e. the sum of the absolute values of the eigenvalues of a signed graph Σ . Many authors have constructed non-cospectral equienergetic graphs. In this paper, we established the spectra of heterogeneous unbalanced net-regular signed complete graphs. Then, we give a method to construct an infinite pairs of unbalanced, non-cospectral, equienergetic, net-regular signed graphs of the same order.

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1 Introduction

We consider the graph G is a simple undirected graph without loops and multiple edges with n vertices and m edges. A signed graph (or sigraph) is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is a graph called the underlying graph of Σ and $\sigma : E \rightarrow \{+1, -1\}$, called signing (or a signature), is a function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. If all the edges of Σ are assigned either $+$ or $-$ sign then Σ is known as homogeneous signed graph and heterogeneous otherwise. The sign of a cycle in a sigraph is the product

of the signs of its edges. Thus a cycle is positive if it contains an even number of negative edges. A signed graph is said to be balanced (or cycle balanced) if all of its cycles are positive.

Let $\Sigma = (G, \sigma)$ be a signed graph with vertex set $V(\Sigma) = V(G) = \{v_1, v_2, \dots, v_n\}$, then its adjacency matrix is defined as a square matrix $A(\Sigma) = [a_{ij}]_{n \times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if the edge between } v_i \text{ and } v_j \text{ is positive.} \\ -1, & \text{if the edge between } v_i \text{ and } v_j \text{ is negative.} \\ 0, & \text{otherwise.} \end{cases}$$

The negation of a signed graph $\Sigma = (G, \sigma)$, denoted by $\eta(\Sigma) = (G, \sigma)$ is the same graph with all signs reversed. The adjacency matrices are related by $A(-\Sigma) = -A(\Sigma)$.

The characteristic polynomial of the signed graph Σ is defined as

$$\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma)) = \sum_{i=1}^n c_i \lambda^{n-i}$$

where I is an identity matrix of order n .

The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$ are called the eigenvalues of signed graph Σ . If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and their multiplicities are $m_1, m_2, \dots, m_n$, then the spectrum of Σ is

$$Sp(\Sigma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \dots & \lambda_n \\ m_1 & m_2 & \dots & \dots & m_n \end{pmatrix}.$$

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion for balance in signed graph is given by B.D.Acharya as follows:

Theorem 1.1: [1] A signed graph is balanced if and only if it is cospectral with the underlying graph.

The energy of a signed graph [5] is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix $A(\Sigma)$ of a signed graph Σ , that is, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of Σ , then

$$\varepsilon(\Sigma) = \sum_{i=1}^n |\lambda_i|.$$

Two signed graphs Σ_1 and Σ_2 are said to be equienergetic if $\varepsilon(\Sigma_1) = \varepsilon(\Sigma_2)$.

Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of a signed graph Σ and $d_i^+(d_i^-)$ be the number of positive(negative) edges incident with v_i . However, in signed graph Σ , the degree of v_i is defined as $sdeg(v_i) = d(v_i) = d_i^+ + d_i^-$. The net degree of a vertex v_i of a signed graph Σ is $d(v_i) = d_i^+ - d_i^-$. A signed graph Σ is said to be net-regular[9] of degree k if all its vertices have same net-degree equal to k i.e. $k = d(v_i) = d_i^+ - d_i^-$. It is clear that the net-regularity of a signed graph can be either positive, negative or zero.

In literature, the following results are available to find the eigenvalues of signed graphs.

Lemma 1.2: [5] The signed paths $P_n^{(r)}$, where r is the number of negative edges and $0 \leq r \leq n-1$, have the eigenvalues(independent of r) given by

$$\lambda_j = 2 \cos \frac{\pi j}{n+1}, j = 1, 2, \dots, n.$$

Lemma 1.3: [14, 5] The eigenvalues λ_j of signed cycles $C_n^{(r)}$ and $0 \leq r \leq n$ are given by

$$\lambda_j = 2 \cos \frac{(2j - [r])\pi}{n}, j = 1, 2, \dots, n$$

where r is the number of negative edges and $[r] = 0$ if r is even, $[r] = 1$ if r is odd.

Spectra of graphs is well documented in[4] and signed graphs is discussed in[5, 6, 7, 14]. For standard terminology and notations in graph theory we follow D.B.West[18] and for signed graphs we follow T. Zaslavsky[20].

There are several papers on construction of equienergetic graphs (see[2, 3, 10, 11, 12, 13, 15, 16, 17, 19]). The objective of this paper is to construct non-cospectral equienergetic signed graphs of same order which are unbalanced.

The problem of construction of equienergetic signed graphs is an open problem[5]. The main aim of this paper is to construct such signed graphs. First, we establish a spectrum for one class of net-regular signed complete graphs in order to find the spectrum for unbalanced signed graphs on $2n$ vertices. Then we construct an infinite family of unbalanced non-cospectral and equienergetic net-regular signed graphs on $2n$ vertices. It is also shown that Coulson's Integral formula for graph energy still holds for signed graphs.

2 Main Results

Here we established a spectrum for one class of heterogeneous net-regular signed complete graphs.

Definition 2.1: Let C_n be a cycle on n vertices and $\overline{C}_n$ be its complement where $n \geq 4$. Consider a signed graph such that $\sigma(u, v) = 1$ if u is adjacent to v in C_n and $\sigma(u, v) = -1$ if u is adjacent to v in $\overline{C}_n$. The resultant signed graph is an unbalanced net-regular signed complete graph and we denote it as K_n^{net} where $n \geq 4$.

Lemma 2.2:[4] Let C_n be a cycle on n vertices and $\overline{C}_n$ be its complement. Spectrum of C_n is

$$SpC_n = \begin{pmatrix} 2 & 2\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, \dots, n-1.$$

Spectrum of $\overline{C}_n$ is

$$Sp\overline{C}_n = \begin{pmatrix} n-3 & -1-2\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, \dots, n-1.$$

The following result gives the spectrum of unbalanced net-regular signed complete graph K_n^{net} .

Theorem 2.3 : The spectrum of heterogeneous unbalanced signed complete graph (K_n^{net}) is

$$Sp(K_n^{net}) = \begin{pmatrix} 5-n & 1+4\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, \dots, n-1.$$

Proof : Adjacency matrix of K_n^{net} can be written as

$$A(K_n^{net}) = A(C_n) + A(\eta(\overline{C}_n))$$

Hence, we can write

$$Sp(K_n^{net}) = Sp(C_n) + Sp(\eta(\overline{C}_n))$$

By Lemma 2.2, the Spectrum of K_n^{net} is

$$Sp(K_n^{net}) = \begin{pmatrix} 2 & 2\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 3-n & 1+2\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, \dots, n-1.$$

$$Sp(K_n^{net}) = \left(\begin{array}{cc} 5-n & 1+4\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

□

3 Equienergetic Net-regular Signed Graphs

Recently, graph energy is generalized to signed graphs. Here we note that well known Coulson's Integral formula for graphs is valid for signed graphs for calculating the energy and proof of the theorem is analogous to graphs[8].

Theorem 3.1: If Σ is a signed graph then the energy of signed graph Σ is

$$\varepsilon(\Sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[n - \frac{i\lambda\phi'(i\lambda)}{\phi(i\lambda)} \right] d\lambda.$$

Example 3.2: Let C_4^- be a signed cycle with odd number of negative edges. Then the characteristic polynomial of $A(C_4^-)$ is $\phi(\lambda) = \lambda^4 - 4\lambda^2 + 4$ and eigenvalues are $\sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}$ and hence $\varepsilon(C_4^-) = 4\sqrt{2}$.

By Coulson's integral formula,

$$\begin{aligned} \varepsilon(C_4^-) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[4 - \frac{i\lambda[4(i\lambda)^3 - 8(i\lambda)]}{(i\lambda)^4 - 4(i\lambda)^2 + 4} \right] d\lambda \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[4 - \frac{(4\lambda^4 + 8\lambda)}{(\lambda^4 + 4\lambda^2 + 4)} \right] d\lambda \\ &= \frac{8}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\lambda^2 + 2} \right] d\lambda \\ &= 4\sqrt{2} \end{aligned}$$

□

Now, we shall introduce the following operation to construct heterogeneous, net-regular signed graphs Σ and Σ^* .

Construction 3.3: Let Σ_1 and Σ_2 be two copies of K_n^{net} with vertex sets $V(\Sigma_1) = \{v_1, v_2, \dots, v_n\}$ and $V(\Sigma_2) = \{u_1, u_2, \dots, u_n\}$. Let Σ be a signed graph obtained from the union of signed graphs Σ_1 and Σ_2 by adding $\{v_i u_i : i \in \{1, 2, \dots, n\}\}$ with positive edges. The derived signed graph Σ is an unbalanced, k net-regular signed graph where its underlying graph is with regularity r .

Construction 3.4: To get net-regular signed graph Σ^* , assign positive and negative signs to the edges of the complement of underlying graph of Σ in a circulant way i.e. $(+1, -1, -1, \dots, -1, +1)$ to $\{v_i u_j : i, j \in \{1, 2, \dots, n\}\}$. Then Σ^* will be a $(k-1)$ net-regular unbalanced signed bipartite graph.

The following well known result is used for the investigation.

Lemma 3.5 : [4] Let $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Theorem 3.6: Let Σ and Σ^* be two net-regular signed graphs as defined above on $2n$ vertices for $n \geq 4$. Then the spectrum of Σ and Σ^* are

$$(i) Sp(\Sigma) = \left(\begin{array}{cc} 6-n & 4-n \\ 1 & 1 \end{array} \begin{array}{cc} 4 \cos(\frac{2\pi j}{n}) & 2 + 4 \cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

$$(ii) Sp(\Sigma^*) = \left(\begin{array}{cc} -(5-n) & -(1 + 4 \cos(\frac{2\pi j}{n})) \\ 1 & 1 \end{array} \begin{array}{cc} 5-n & 1 + 4 \cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

Proof : (i) Adjacency matrix of Σ can be written as

$$A(\Sigma) = \begin{pmatrix} K_n^{net} & I_n \\ I_n & K_n^{net} \end{pmatrix}.$$

where I_n is the identity matrix of order n .

By Theorem 2.3,

Spectrum $(K_n^{net} + I_n)$ is

$$Sp(K_n^{net} + I_n) = \left(\begin{array}{cc} 6-n & 2 + 4 \cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

Spectrum $(K_n^{net} - I_n)$ is

$$Sp(K_n^{net} - I_n) = \left(\begin{array}{cc} 4-n & 4 \cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

By Lemma 3.5, Spectrum of Σ is the union of $Sp(K_n^{net} + I_n)$ and $Sp(K_n^{net} - I_n)$.

Hence the spectrum of Σ is

$$Sp(\Sigma) = \left(\begin{array}{cc} 6-n & 4-n \\ 1 & 1 \end{array} \begin{array}{cc} 4 \cos(\frac{2\pi j}{n}) & 2 + 4 \cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

(ii) W.L.O.G. Adjacency matrix of Σ^* can be written as

$$A(\Sigma^*) = \begin{pmatrix} 0 & K_n^{net} \\ K_n^{net} & 0 \end{pmatrix}.$$

By Lemma 3.5, Spectrum of Σ^* is the union of $\text{Sp}(K_n^{net})$ and $\text{Sp}(-K_n^{net})$.

We get the spectrum of Σ^* as

$$Sp(\Sigma^*) = \left(\begin{array}{cc} -(5-n) & -(1+4\cos(\frac{2\pi j}{n})) \\ 1 & 1 \end{array} \begin{array}{cc} (5-n) & (1+4\cos(\frac{2\pi j}{n})) \\ 1 & 1 \end{array} \right) : j = 1, \dots, n-1.$$

□

Theorem 3.7: Let Σ and Σ^* be two net-regular, unbalanced, heterogeneous signed graphs as defined above. Then Σ and Σ^* are non-cospectral and equienergetic on $2n$ vertices for $n \geq 4$.

Proof : From Theorem 3.6, the spectrum of Σ and Σ^* are not identical implies that Σ and Σ^* are not cospectral.

By Theorem 3.6, we can find the energy of Σ and Σ^* as follows.

(i) Energy of Σ is

$$\begin{aligned} \varepsilon(\Sigma) &= |(6-n)| + |(1)| + |(4-n)| + |(1)| + \sum_{j=1}^{n-1} |2+4\cos(\frac{2\pi j}{n})| + \sum_{j=1}^{n-1} |4\cos(\frac{2\pi j}{n})|. \\ &= 2[|(5-n)|] + 2[\sum_{j=1}^{n-1} |1+4\cos(\frac{2\pi j}{n})|]. \end{aligned}$$

(ii) Since the spectrum of Σ^* is symmetric with respect to zero, Energy of Σ^* is

$$\varepsilon(\Sigma^*) = 2[|(5-n)|] + 2[\sum_{j=1}^{n-1} |1+4\cos(\frac{2\pi j}{n})|].$$

From (i) and (ii), $\varepsilon(\Sigma) = \varepsilon(\Sigma^*)$.

This completes the proof. □

Remark 3.8 : From Theorem 3.6 and 3.7, we can see that there exists an infinite pairs of non-cospectral and equienergetic, unbalanced, net-regular signed graphs on $2n$ vertices for $n \geq 4$.

We conclude this paper by posing a problem that “Is there any other method to construct unbalanced, non-cospectral, equienergetic signed graphs on n vertices?”

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