A Cardinal Function on the Category of Metric Spaces

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Abstract

In the following text in the metric spaces category we introduce a topologically invariant cardinal function, Θ (clearly Θ is a tool to classify metric spaces). For this aim in metric space X we consider cone metrics $(X, \mathbf{H}, P, \rho)$ such that $\mathbf{H}$ is a Hilbert space, image of ρ has nonempty interior, and this cone metric induces the original metric topology on X . We prove that for all sufficiently large cardinal numbers α , there exists a metric space (X, d) with $\Theta(X, d) = \alpha$.

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1 Introduction

As it has been mentioned in several texts, cone metric spaces in the following form has been introduced for the first time in [9] as a generalization of metric

spaces (e.g. see [8]). Several papers has been published regarding the matter since 2007. A large number of these papers deal with fixed point theorems (e.g. see [9], [1], [6]), however there are texts deal with the other properties (e.g., [2]) amongst these texts some authors try to study metrizable or interaction between metric spaces and cone metric spaces ([5], [3], [4]).

In this text $\mathbf{R}$ denotes the set of all real numbers, and $\mathbf{N} = \{1, 2, \dots\}$ denotes the set of all natural numbers.

Note. All vector spaces assumed in this text are nonzero real vector spaces.

In (real) norm vector space $(\mathbf{E}, \|\cdot\|)$ we say $P(\subseteq \mathbf{E})$ is a cone if:

- P is a closed nonempty subset of $\mathbf{E}$,
- for all $x, y \in P$ and $\lambda, \mu \geq 0$ we have $\lambda x + \mu y \in P$,
- $P \cap -P = \{0\}$,

and in addition it is a solid cone in $\mathbf{E}$, if $P^\circ \neq \emptyset$.

Suppose P is a cone in norm vector space $\mathbf{E}$. For all $x, y \in \mathbf{E}$ we say $x \leq_P y$ if $y - x \in P$. Obviously in this way $(\mathbf{E}, \leq_P)$ is a partial ordered set. For all $x, y \in \mathbf{E}$ we say $x <_P y$ if $x \leq_P y$ and $x \neq y$, moreover we say $x \ll_P y$ or simply $x \ll y$ if $y - x \in P^\circ$.

We say $(X, \mathbf{E}, P, d)$ is a cone metric space if P is a cone in $\mathbf{E}$ and $d : X \times X \rightarrow P$ for all $a, b, c \in X$ satisfies the following conditions:

- $d(a, b) = 0$ if and only if $a = b$,
- $d(a, b) = d(b, a)$,
- $d(a, b) \leq_P d(a, c) + d(c, b)$.

In the cone metric space $(X, \mathbf{E}, P, d)$ for $\varepsilon \gg 0$ and $x \in \mathbf{E}$ let $B^d(x, \varepsilon)$ or simply $B(x, \varepsilon)$ is $\{z \in \mathbf{E} : d(z, x) \ll \varepsilon\}$. It is easy to see that $\{B(x, \varepsilon) : x \in \mathbf{E}, \varepsilon \gg 0\}$ is a topological basis on X . We consider cone metric space $(X, \mathbf{E}, P, d)$ under topology generated by the above basis on X .

It is well-known that for every (real) Hilbert space $\mathbf{H}$ there exists a nonzero cardinal number α such that $\mathbf{H}$ and $\ell^2(\alpha)$ are isomorphic (as Hilbert spaces), where for nonempty set Γ we have:

$$\ell^2(\Gamma) = \left\{ (x_\lambda)_{\lambda \in \Gamma} \in \mathbf{R}^\Gamma : \sum_{\lambda \in \Gamma} |x_\lambda|^2 < \infty \right\}$$

equipped with inner product $\langle (x_\lambda)_{\lambda \in \Gamma}, (y_\lambda)_{\lambda \in \Gamma} \rangle := \sum_{\lambda \in \Gamma} x_\lambda y_\lambda$ and therefore

$$\text{norm } \|(x_\lambda)_{\lambda \in \Gamma}\| = \left(\sum_{\lambda \in \Gamma} |x_\lambda|^2 \right)^{\frac{1}{2}} \quad (\text{for } (x_\lambda)_{\lambda \in \Gamma}, (y_\lambda)_{\lambda \in \Gamma} \in \ell^2(\Gamma)).$$

We recall that for all nonzero cardinal number α , we have $\alpha = \{\beta \in \text{ON} : \beta < \alpha\}$ where ON is the class of all ordinal numbers (by CN we mean the class of all cardinal numbers which is a proper subclass of ON). We denote the least infinite cardinal number with $\aleph_0$ or ω and the cardinality of $\mathbf{R}$ with c . Also it is well-known that $\ell^2(n)$ can be considered as $\mathbf{R}^n$ with Euclidean norm for $1 \leq n < \omega$.

2 First steps

In this section we get ready to define Hilbert-cone metric order of a metric space (X, d) .

Lemma 2.1 *In norm vector space $(\mathbf{F}, \|\cdot\|)$ if V is a nonvoid open subset of $\mathbf{F}$, then $\text{card}(V) = \text{card}(\mathbf{F}) = \max(c, \alpha)$, where $\alpha = \dim_{\mathbf{R}}(\mathbf{F})$ (α is the cardinality of a Hamel basis of $\mathbf{F}$ over $\mathbf{R}$).*

Proof. For $r > 0$ let $U_r = \{x \in \mathbf{F} : \|x\| < r\}$. For $s > 0$, $\text{card}(U_r) = \text{card}(U_s)$, since $\varphi : U_r \rightarrow U_s$ with $\varphi(x) = \frac{s}{r}x$ is bijective. On the other hand $\mathbf{F} = \bigcup\{U_n : n \in \mathbf{N}\}$, which leads to (since for all $n \in \mathbf{N}$ we have $\text{card}(U_1) = \text{card}(U_n)$):

$$\text{card}(U_1) \leq \text{card}(\mathbf{F}) \leq \aleph_0 \text{card}(U_1).$$

Using $\aleph_0 < \text{card}(\mathbf{R}) \leq \text{card}(\mathbf{F}) \leq \aleph_0 \text{card}(U_1)$ we have $\aleph_0 \text{card}(U_1) = \text{card}(U_1)$. By $\text{card}(U_1) \leq \text{card}(\mathbf{F}) \leq \aleph_0 \text{card}(U_1) = \text{card}(U_1)$ we have:

$$\text{card}(\mathbf{F}) = \text{card}(U_1).$$

Suppose V is a nonvoid open subset of $\mathbf{F}$, there exist $z \in V$ and $r > 0$ such that $z + U_r \subseteq V$. Since $\eta : U_1 \rightarrow V$ with $\eta(x) = rz + x$ is injective, we have:

$$\text{card}(\mathbf{F}) \geq \text{card}(V) \geq \text{card}(U_1) = \text{card}(\mathbf{F}).$$

Therefore $\text{card}(V) = \text{card}(\mathbf{F}) = \max(c, \alpha)$.

Theorem 2.2 *In cone metric space $(X, \mathbf{E}, P, d)$ if $d(X \times X)^\circ \neq \emptyset$, then:*

$$\text{card}(X) \geq \text{card}(\mathbf{E}).$$

Proof. Using Lemma 2.1, $\text{card}(d(X \times X)^\circ) = \text{card}(\mathbf{E})$. Moreover

$$\text{card}(X \times X) \geq \text{card}(d(X \times X)) \geq \text{card}(d(X \times X)^\circ) = \text{card}(\mathbf{E})$$

and X is infinite, therefore $\text{card}(X) = \text{card}(X \times X) \geq \text{card}(\mathbf{E})$.

Using Theorem 2.2, in cone metric space $(X, \mathbf{E}, P, d)$ if $\text{card}(X) < \text{card}(\mathbf{E})$, then $d(X \times X)^\circ = \emptyset$.

Lemma 2.3 For $\alpha \geq c$ we have $\text{card}(\ell^2(\alpha)) = \alpha$.

Proof. Consider $\alpha \geq c$, for all $(x_\theta)_{\theta \in \alpha} \in \ell^2(\alpha)$ there exists a countable subset D of α such that for all $\theta \in \alpha \setminus D$ we have $x_\theta = 0$. Therefore $\text{card}(\ell^2(\alpha)) \leq \text{card}(\{(D, (x_\theta)_{\theta \in D}) : D \text{ is a countable subset of } \alpha \text{ and } x_\theta \in \mathbf{R} \text{ for all } \alpha \in D\})$, which leads to $\text{card}(\ell^2(\alpha)) \leq \text{card}(\{(f, (x_n)_{n \in \mathbf{N}}) : f : \mathbf{N} \rightarrow \alpha \text{ is an injection and } x_n \in \mathbf{R} \text{ for all } n \in \mathbf{N}\})$ and:

$$\text{card}(\ell^2(\alpha)) \leq \text{card}(\alpha^{\mathbf{N}} \times \mathbf{R}^{\mathbf{N}}) = \alpha^{\aleph_0} c^{\aleph_0} = \alpha$$

therefore $\text{card}(\ell^2(\alpha)) = \alpha$.

Definition 2.4 In metric space (X, d) let

$$\Theta(X, d) := \sup(\{0\} \cup \{\alpha \in \text{CN} : \text{there exists cone metric space } (X, \ell^2(\alpha), P, \rho) \text{ which induces the same topology as metric topology on } (X, d) \text{ and } \rho(X \times X)^\circ \neq \emptyset\}).$$

Using Theorem 2.2 and Lemma 2.3, $\Theta(X, d) (\leq \text{card}(X))$ exists. We call $\Theta(X, d)$, cone metric order of (X, d) with respect to Hilbert spaces, or simply Hilbert-cone metric order of (X, d) .

Note. Naturally Θ gives a classification of metric spaces. Instead of classifying metric spaces regarding collection $\{\ell^2(\alpha) : \alpha \in \text{CN} \setminus \{0\}\}$, one may consider classification regarding $\mathcal{F}_p = \{\ell^p(\alpha) : \alpha \in \text{CN} \setminus \{0\}\}$ for $1 \leq p \leq +\infty$, so let

$$\Theta_p(X, d) := \sup(\{0\} \cup \{\alpha \in \text{CN} : \text{there exists cone metric space } (X, \ell^p(\alpha), P, \rho) \text{ which induces the same topology as metric topology on } (X, d) \text{ and } \rho(X \times X)^\circ \neq \emptyset\}).$$

And call $\Theta_p(X, d)$, cone metric order of (X, d) with respect to $\mathcal{F}_p$, or simply $\mathcal{F}_p$ -cone metric order of (X, d) . Now one may be interested to study the relation between $\Theta_p(X, d)$ and $\Theta_q(X, d)$ for $p, q \geq 1$.

3 Towards main theorem: A useful example

The main aim of this section is to prove that for $\beta \geq c$ there exists metric space (Z, ψ) with $\Theta(Z, \psi) = \beta$.

In this section suppose $\alpha \in \text{CN}$ is a nonzero cardinal number and $\mathcal{G} = (\mathcal{G}_\theta)_{\theta \in \alpha}$ where $\mathcal{G}_0 = 1$ and $\mathcal{G}_\theta = 0$ for $\theta \neq 0$. Moreover set

$$P = \{(x_\lambda)_{\lambda \in \alpha} \in \ell^2(\alpha) : x_0 \geq 0 \wedge (\forall \beta \in \alpha \setminus \{0\} (|x_\beta| \leq x_0))\},$$

$$Q = \{(x_\theta)_{\theta \in \alpha} \in P : \sup\{|x_\theta| : \theta \neq 0\} < \min(x_0, 1)\},$$

$$X = \ell^2(Q) \setminus \{0\}.$$

Also for $v \in Q$, suppose $u_v = (\delta_\theta^v)_{\theta \in Q}$ with $\delta_v^v = 1$ and $\delta_\theta^v = 0$ for $\theta \neq v$.

Now define $d : X \times X \rightarrow \mathbf{R}$ with:

$$d(x, y) = \begin{cases} \min(|\lambda - \mu|, 1) & v \in Q, \lambda, \mu \in \mathbf{R}, x = \lambda u_v, y = \mu u_v \\ 0 & x = y \\ 4 & \text{otherwise} \end{cases}$$

and $\rho : X \times X \rightarrow P$ with:

$$\rho(x, y) = \begin{cases} d(x, y)v & v \in Q, x, y \in \mathbf{R}u_v \\ 0 & x = y \\ 4\mathcal{G} & \text{otherwise} \end{cases}$$

Lemma 3.1 *The set P is a solid cone in $\ell^2(\alpha)$.*

Proof. We prove the lemma step by step:

- For $\beta \in \alpha$ the map $\varphi_\beta : \ell^2(\alpha) \rightarrow \mathbf{R}^2$ with $\varphi_\beta((x_\lambda)_{\lambda \in \alpha}) = (x_0, x_\beta)$ is continuous, since $|x_\beta| \leq \|(x_\lambda)_{\lambda \in \alpha}\|$ and $|x_0| \leq \|(x_\lambda)_{\lambda \in \alpha}\|$. Therefore

$$P = \bigcap \{\varphi_\beta^{-1}(\{(x, y) \in \mathbf{R}^2 : |y| \leq x\}) : \beta \in \alpha\}$$

and P is close. Moreover clearly $P \cap -P = \{0\}$.

- Suppose $x = (x_\lambda)_{\lambda \in \alpha}, y = (y_\lambda)_{\lambda \in \alpha} \in P$ and $r, s > 0$. We have:

$$\begin{aligned} x, y \in P &\Rightarrow (x_0 \geq 0 \wedge y_0 \geq 0 \wedge (\forall \beta \in \alpha (|x_\beta| \leq x_0 \wedge |y_\beta| \leq y_0))) \\ &\Rightarrow (rx_0 + sy_0 \geq 0 \wedge (\forall \beta \in \alpha (|rx_\beta + sy_\beta| \leq rx_0 + sy_0))) \\ &\Rightarrow rx + sy \in P. \end{aligned}$$

- If $z = 2\mathcal{G} = (z_\lambda)_{\lambda \in \alpha}$, then $z \in P$ and for $x = (x_\lambda)_{\lambda \in \alpha} \in \ell^2(\alpha)$ we have:

$$\begin{aligned} \|z - x\| < 1 &\Rightarrow |z_0 - x_0| < 1 \wedge (\forall \beta \in \alpha |z_\beta - x_\beta| < 1) \\ &\Rightarrow |2 - x_0| < 1 \wedge (\forall \beta \in \alpha \setminus \{0\} |x_\beta| < 1) \\ &\Rightarrow x_0 > 1 \wedge (\forall \beta \in \alpha \setminus \{0\} |x_\beta| < 1) \\ &\Rightarrow x_0 \geq 0 \wedge (\forall \beta \in \alpha \setminus \{0\} |x_\beta| < x_0) \\ &\Rightarrow x_0 \geq 0 \wedge (\forall \beta \in \alpha |x_\beta| \leq x_0) \\ &\Rightarrow x \in P \end{aligned}$$

Therefore $\{x \in \ell^2(\alpha) : \|z - x\| < 1\} \subseteq P$ and $z \in P^\circ$.

Note 3.2 *For each $(x_\theta)_{\theta \in \alpha} \in \ell^2(\alpha)$ with $\sup\{|x_\theta| : \theta \in \alpha\} < 1$ we have $(x_\theta)_{\theta \in \alpha} <_P 2\mathcal{G}$, i.e. $2\mathcal{G} - (x_\theta)_{\theta \in \alpha} \in P$. Moreover $\mathcal{G} \gg 0$ and $(x_\theta)_{\theta \in \alpha} <_P 2\mathcal{G}$ leads us to $3\mathcal{G} \gg (x_\theta)_{\theta \in \alpha}$.*

Lemma 3.3 $(X, \ell^2(\alpha), P, \rho)$ is a cone metric space.

Proof. It is clear that (X, d) is a metric space. Suppose $x, y, z \in X$. If $x \neq y$ and for all $v \in Q$, $x \notin Ru_v$ or $y \notin Ru_v$, then $\rho(x, y) = 4\mathcal{G} \in P$, on the other hand if there exists $v \in Q$, such that $x, y \in Ru_v$, then $d(x, y) \geq 0$ and $v \in P$, thus $\rho(x, y) = d(x, y)v \in P$.

Moreover (note to the fact that $0 \notin Q$):

$$\begin{aligned} \rho(x, y) = 0 &\Leftrightarrow x = y \vee (\exists v \in Q (x, y \in Ru_v \wedge 0 = \rho(x, y) = d(x, y)v)) \\ &\Leftrightarrow x = y \vee d(x, y) = 0 \\ &\Leftrightarrow x = y \end{aligned}$$

It is clear that $\rho(x, y) = \rho(y, x)$.

We complete the proof, using the following cases:

- For $v \in Q$ if $x, y, z \in Ru_v$, then $d(x, y) + d(y, z) - d(x, z) \geq 0$, thus:

$$\rho(x, y) + \rho(y, z) - \rho(x, z) = (d(x, y) + d(y, z) - d(x, z))v \in P.$$

- For $v \in Q$ if $x, y \in Ru_v$ and $z \notin Ru_v$, then:

$$\rho(x, y) + \rho(y, z) - \rho(x, z) = \rho(x, y) + 4\mathcal{G} - 4\mathcal{G} = \rho(x, y) \in P.$$

- For $v \in Q$ if $x, z \in Ru_v$ and $y \notin Ru_v$, then using Note 3.2, $2\mathcal{G} - d(x, z)v \in P$, since $d(x, z) \leq 1$. Thus $\rho(x, y) + \rho(y, z) - \rho(x, z) = 8\mathcal{G} - \rho(x, z) = 6\mathcal{G} + (2\mathcal{G} - d(x, z)v) \in P + P \subseteq P$.

- For $v \in Q$ if $y, z \in Ru_v$ and $x \notin Ru_v$, then $\rho(x, y) + \rho(y, z) - \rho(x, z) = 4\mathcal{G} - \rho(y, z) - 4\mathcal{G} = \rho(y, z) \in P$.

- If $x, y, z \notin Ru_v$ for all $v \in P^\circ$, and x, y, z are pairwise distinct, then $\rho(x, y) + \rho(y, z) - \rho(x, z) = 4\mathcal{G} + 4\mathcal{G} - 4\mathcal{G} = 4\mathcal{G} \in P$.

- If $x = y$ or $y = z$, then $\rho(x, y) + \rho(y, z) - \rho(x, z) = 0 \in P$. If $x = z$, then $\rho(x, y) + \rho(y, z) - \rho(x, z) = \rho(x, y) + \rho(y, z) \in P + P \subseteq P$.

Therefore $\rho(x, y) + \rho(y, z) - \rho(x, z) \in P$ and $\rho(x, z) \leq_P \rho(x, y) + \rho(y, z)$.

Lemma 3.4 $\rho(X \times X)^\circ \neq \emptyset$.

Proof. For all $v \in Q$ we have $\rho(u_v, \frac{1}{2}u_v) = \frac{v}{2}$, thus $\frac{1}{2}Q \subseteq \rho(X \times X)$. Moreover for $y = (y_\theta)_{\theta \in \alpha} \in \ell^2(\alpha)$ we have:

$$\begin{aligned} \|y - \mathcal{G}/2\| < 1/4 &\Rightarrow \|2y - \mathcal{G}\| < 1/2 \\ &\Rightarrow \forall \theta \in \alpha |2y_\theta - \mathcal{G}_\theta| < 1/2 \\ &\Rightarrow |2y_0 - 1| < 1/2 \wedge (\forall \theta \in \alpha \setminus \{0\} |y_\theta| < 1/4) \\ &\Rightarrow 1/4 < y_0 < 3/4 \wedge \sup\{|y_\theta| : \theta \neq 0\} \leq 1/4 \\ &\Rightarrow y_0 > 0 \wedge \sup\{|y_\theta| : \theta \neq 0\} < y_0 = \min(y_0, 1) \\ &\Rightarrow y \in Q \end{aligned}$$

Using $\{y \in \ell^2(\alpha) : \|y - \frac{g}{2}\| < \frac{1}{4}\} \subseteq Q$, we have:

$$\left\{y \in \ell^2(\alpha) : \left\|y - \frac{g}{4}\right\| < \frac{1}{8}\right\} \subseteq \frac{1}{2}Q \subseteq \rho(X \times X)$$

which leads to the desired result.

Lemma 3.5 *Induced topology of metric d on X and induced topology of cone metric ρ on X are the same.*

Proof. For all $x \in X \setminus \bigcup\{\mathbf{R}u_v : v \in Q\}$, we have $\{y \in X : \rho(x, y) <_{\mathbf{P}} 4\mathcal{G}\} = \{x\}$. Therefore $\{y \in X : \rho(x, y) \ll 4\mathcal{G}\} = \{y \in X : d(x, y) < 1\} = \{x\}$. Now suppose $x = \lambda u_v \in X$ for $\lambda \in \mathbf{R} \setminus \{0\}$ and $v = (t_\theta)_{\theta \in \alpha} \in Q$. If $\varepsilon \gg 0$, then there exists real number $r < \min(t_0 - \sup\{|t_\theta| : \theta \neq 0\}, 1)$ such that $0 < r$ and $0 \ll 6rt_0\mathcal{G} \ll \varepsilon$ ($t_0 > 0$ since $v \in Q$). For all $y \in X \setminus \{x\}$, if $d(x, y) < r$, then $y \in \mathbf{R}u_v \setminus \{0\}$ and $\rho(x, y) = d(x, y)v (\neq 0)$. Moreover $v \ll 6t_0\mathcal{G}$, thus

$$\rho(x, y) = d(x, y)v \ll 6t_0d(x, y)\mathcal{G} \ll 6t_0r\mathcal{G} \ll \varepsilon.$$

Hence:

$$\forall \varepsilon \gg 0 \quad \exists r > 0 \quad \{y \in X : d(x, y) < r\} \subseteq \{y \in X : \rho(x, y) \ll \varepsilon\}.$$

Conversely, if $r > 0$, using $t_0 > 0$, there exists $s > 0$ such that $\frac{s}{t_0} < r$ and $s < 4$. We have $\varepsilon := s\mathcal{G} \gg 0$. For all $y \in X$ if $\rho(x, y) \ll_{\mathbf{P}} s\mathcal{G} <_{\mathbf{P}} 4\mathcal{G}$, then $y \in \mathbf{R}u_v \setminus \{0\}$ and we have:

$$\begin{aligned} \rho(x, y) \ll s\mathcal{G} &\Rightarrow s\mathcal{G} - d(x, y)v \in \mathbf{P}^\circ \subseteq \mathbf{P} \\ &\Rightarrow s - d(x, y)t_0 \geq 0 \\ &\Rightarrow d(x, y) \leq \frac{s}{t_0} < r. \end{aligned}$$

Hence:

$$\forall r > 0 \quad \exists \varepsilon \gg 0 \quad \{y \in X : \rho(x, y) \ll \varepsilon\} \subseteq \{y \in X : d(x, y) < r\},$$

which completes the proof.

Corollary 3.6 *(X, d) is a disconnected space with $\Theta(X, d) \geq \alpha$.*

Proof. Use Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Lemma 3.7 *If $\alpha \geq c$, then $\text{card}(X) = \alpha$.*

Proof. Using the proof of Lemma 3.4 we have $Q^\circ \neq \emptyset$. So by Lemma 2.1, $\text{card}(Q^\circ) = \text{card}(\ell^2(\alpha))$, which leads to $\text{card}(Q) = \text{card}(\ell^2(\alpha)) = \alpha$. Therefore $\text{card}(X) = \text{card}(\ell^2(Q)) = \text{card}(\ell^2(\alpha)) = \alpha$.

Lemma 3.8 *If $\alpha \geq c$, then $\Theta(X, d) = \alpha$.*

Proof. Using Lemma 3.7, Corollary 3.6 and $\Theta(X, d) \leq \text{card}(X)$, we have $\Theta(X, d) = \alpha$.

Theorem 3.9 (Main Theorem) *For $\beta \geq c$ there exists (disconnected) metric space (Z, ψ) with $\Theta(Z, \psi) = \beta$.*

Proof. Use Lemma 3.8.

4 Products of metric spaces and Θ function

In this section for two metric spaces (X, d) and (Y, k) , consider $X \times Y$ under metric $\sigma_{(d,k)}$, where $\sigma_{(d,k)}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + k(y_1, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Also in two norm vector spaces $(\mathbf{E}, \rho)$ and $(\mathbf{F}, \mu)$, consider norm vector space $(\mathbf{E} \times \mathbf{F}, \sigma_{(\rho,\mu)})$, where $\sigma_{(\rho,\mu)}(x, y) = \rho(x) + \mu(y)$ for all $(x, y) \in \mathbf{E} \times \mathbf{F}$.

4.1 Product of two cone metric space

For $f : A \rightarrow B$ and $g : C \rightarrow D$ define $f \times g : A \times C \rightarrow B \times D$ with $(f \times g)(x, y) = (f(x), g(y))$. In the following we will prove $\Theta(X \times Y, \sigma_{(d,k)}) \geq \Theta(X, d) + \Theta(Y, k)$ for two metric spaces (X, d) and (Y, k) .

Lemma 4.1 *If P is a cone in norm vector space $\mathbf{E}$ and Q is a cone in norm vector space $\mathbf{F}$, then $P \times Q$ is a cone in norm vector space $\mathbf{E} \times \mathbf{F}$. Moreover P and Q are solid if and only if $P \times Q$ is solid.*

Proof. It is clear that $-(P \times Q) \cap (P \times Q) = (-P \cap P) \times (-Q \cap Q) = \emptyset \times \emptyset = \emptyset$. On the other hand for $(x, y), (z, w) \in P \times Q$ and $\lambda, \mu \geq 0$ we have $\lambda x + \mu z \in P$ and $\lambda y + \mu w \in Q$, therefore $\lambda(x, y) + \mu(z, w) = (\lambda x + \mu z, \lambda y + \mu w) \in P \times Q$. $P \times Q$ is a closed nonempty subset of $\mathbf{E} \times \mathbf{F}$, since P is a nonempty closed subset of $\mathbf{E}$ and Q is a nonempty closed subset of $\mathbf{F}$. Thus $P \times Q$ is a cone in $\mathbf{E} \times \mathbf{F}$.

Note to the fact that $(A \times B)^\circ = A^\circ \times B^\circ$, to complete the proof.

Lemma 4.2 *If $(X, \mathbf{E}, P, d)$ and $(Y, \mathbf{F}, Q, k)$ are cone metric spaces, then $(X \times Y, \mathbf{E} \times \mathbf{F}, P \times Q, d \times k)$ is a cone metric space, where $(d \times k)((x_1, y_1), (x_2, y_2)) = (d(x_1, x_2), k(y_1, y_2))$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.*

Proof. Use Lemma 4.1.

Theorem 4.3 (Product of cone metric spaces) *Consider two cone metric spaces $(X, \mathbf{E}, P, d)$ and $(Y, \mathbf{F}, Q, k)$. Cone metric topology on $(X \times Y, \mathbf{E} \times \mathbf{F}, P \times Q, d \times k)$ is product topology on $X \times Y$ when X is considered under cone metric topology $(X, \mathbf{E}, P, d)$ and Y considered under cone metric topology $(Y, \mathbf{F}, Q, k)$.*

Proof. Note to the fact that for $(\varrho_1, \varrho_2) \in \mathbf{E} \times \mathbf{F}$ we have $(0, 0) \ll_{P \times Q} (\varrho_1, \varrho_2)$ if and only if $0 \ll_P \varrho_1$ and $0 \ll_Q \varrho_2$. Thus if $0 \ll_P \varepsilon_1$ and $0 \ll_Q \varepsilon_2$ and $(x, y) \in \mathbf{E} \times \mathbf{F}$, then we have:

$$\begin{aligned} \{(z, w) \in \mathbf{E} \times \mathbf{F} : (d \times k)((z, w), (x, y)) \ll_{P \times Q} (\varepsilon_1, \varepsilon_2)\} \\ = \{z \in \mathbf{E} : d(z, x) \ll_P \varepsilon_1\} \times \{w \in \mathbf{F} : k(w, y) \ll_Q \varepsilon_2\}, \end{aligned}$$

which leads to the desired result.

Remark 4.4 For nonzero cardinal numbers $\alpha, \beta \in \mathbf{CN}$, $\ell^2(\alpha) \times \ell^2(\beta)$ and $\ell^2(\alpha + \beta)$ are isomorphic.

Corollary 4.5 If (X, d) and (Y, k) are metric spaces and $\Theta(X, d) \geq \alpha$ and $\Theta(Y, k) \geq \beta$, then $\Theta(X \times Y, \sigma_{(d, k)}) \geq \alpha + \beta$.

Proof. Use Theorem 4.3 and Remark 4.4.

Theorem 4.6 If (X, d) and (Y, k) are metric spaces, then:

$$\Theta(X \times Y, \sigma_{(d, k)}) \geq \Theta(X, d) + \Theta(Y, k).$$

Proof. Use Corollary 4.5.

Note 4.7 If $(X, \mathbf{E}, P, d)$ and $(Y, \mathbf{E}, P, k)$ are cone metric spaces and:

$$\sigma_{(d, k)}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + k(y_1, y_2) \quad ((x_1, y_1), (x_2, y_2) \in X \times Y),$$

then $(X \times Y, \mathbf{E}, P, \sigma_{(d, k)})$ is a cone metric space.

4.2 More details on $\mathbf{R}^n$

For $n \in \mathbf{N}$ suppose E_n is usual metric on $\mathbf{R}^n$ induced from its Euclidean norm. Using Theorem 4.6 two metric spaces $(\mathbf{R}^{n+m}, \sigma_{(E_n, E_m)})$ and $(\mathbf{R}^{n+m}, E_{n+m})$ are homeomorph, hence we have $\Theta(\mathbf{R}^n, E_n) \geq n\Theta(\mathbf{R}, E_1) \geq 1$. Here we want to prove $\Theta(\mathbf{R}^n, E_n) \geq n$ directly.

Note 4.8 For nonzero $n \in \omega$ let:

$$P = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \geq 0, \dots, x_n \geq 0\},$$

then P is a solid cone in $\ell^2(n) = \mathbf{R}^n$ with Euclidean norm.

Theorem 4.9 For nonzero $n \in \omega$, consider $\mathbf{R}^n$ under its usual metric induced from Euclidean norm on $\mathbf{R}^n$, we denote this metric on $\mathbf{R}^n$ with E_n . Then $\Theta(\mathbf{R}^n, E_n) \geq n$.

Proof. Let $P = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}$ as in Note 4.8. Define $d : \mathbf{R}^n \times \mathbf{R}^n \rightarrow P$ with $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = (|x_1 - y_1|, \dots, |x_n - y_n|)$. In cone metric space $(\mathbf{R}^n, \mathbf{R}^n, P, d)$ we have $d(\mathbf{R}^n \times \{x\}) = P$ and $d(\mathbf{R}^n \times \{x\})^\circ = P^\circ \neq \emptyset$ for all $x \in \mathbf{R}^n$. Moreover for all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \gg 0$ and $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we have $\varepsilon_1, \dots, \varepsilon_n > 0$ and:

$$\begin{aligned} \{y \in \mathbf{R}^n : d(x, y) \ll \varepsilon\} &= \{(y_1, \dots, y_n) \in \mathbf{R}^n : (\varepsilon_1 - |x_1 - y_1|, \dots, \varepsilon_n - |x_n - y_n|) \gg 0\} \\ &= \{(y_1, \dots, y_n) \in \mathbf{R}^n : |x_1 - y_1| < \varepsilon_1, \dots, |x_n - y_n| < \varepsilon_n\} \\ &= (x_1 - \varepsilon_1, x_1 + \varepsilon_1) \times \dots \times (x_n - \varepsilon_n, x_n + \varepsilon_n) \end{aligned}$$

Therefore $B^d(x, \varepsilon)$ is an open subset of $(\mathbf{R}^n, E_n)$. Moreover $\{(x_1 - \mu, x_1 + \mu) \times \dots \times (x_n - \mu, x_n + \mu) : x_1, \dots, x_n \in \mathbf{R}, \mu > 0\} (= \{B^d(x, (\mu, \dots, \mu)) : \mu > 0, x \in \mathbf{R}^n\})$ is a topological basis for $(\mathbf{R}^n, E_n)$. Therefore induced topology from metric space $(\mathbf{R}^n, E_n)$ and cone metric space $(\mathbf{R}^n, \mathbf{R}^n, P, d)$ ($= (\mathbf{R}^n, \ell^2(n), P, d)$) are coincide. Hence $\Theta(\mathbf{R}^n, E_n) \geq n$.

5 Some Arising Problems

For each nonzero cardinal number α , suppose $\mathcal{T}_\alpha$ is the class of all metric spaces (X, d) such that $\Theta(X, d) \geq \alpha$. Using Theorem 3.9, $\mathcal{T}_\alpha \neq \emptyset$ for each nonzero $\alpha \in \text{CN}$. Also for $\alpha \geq c$, $\mathcal{T}_{2^\alpha}$ is a proper subclass of $\mathcal{T}_\alpha$.

Also it is well-known [7] that for all connected metric space (X, d) with at least two distinct elements a, b we have $[0, d(a, b)) \subseteq d(X \times \{a\})$, therefore considering cone metric space $(X, \mathbf{R}, [0, +\infty), d)$ leads us to $\Theta(X, d) \geq 1$. In other words:

The class of all connected metric spaces with at least two elements
is a subclass of $\mathcal{T}_1$.

However in Theorem 3.9, we prove that for $\alpha \geq c$ there exists metric space (X, d) with $\Theta(X, d) = \alpha$. Now we have the following problems:

Problem 5.1 For nonzero $\alpha < c$, find a metric space (X, d) with $\Theta(X, d) = \alpha$.

Problem 5.2 For nonzero $\alpha \in \text{CN}$ find a connected metric space (X, d) with $\Theta(X, d) = \alpha$. In particular $\{(X, d) : (X, d) \text{ is a connected metric space with } \Theta(X, d) = \alpha\} : \alpha \text{ is a nonzero cardinal number}\}$ is a meaningful partition of the class of all connected metric spaces?

Problem 5.3 For $\alpha \geq \omega$ in metric space (X, d) suppose $\Theta(X, d) = \alpha$. Is there any cone metric space $(X, \ell^p(\alpha), P, \rho)$ which induces the same topology as metric topology on (X, d) and $\rho(X \times X)^\circ \neq \emptyset$? In other words can we replace “sup” in Definition 2.4 with “max”?

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