Generalised Strongly Prime Ideals in Near Rings

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Abstract

In 1984, Groenewald and Potgieter, generalised results for f -prime ideal in near rings. In this paper, we introduce the notion of strongly f -prime ideal in near rings and generalise some of basic results of strongly prime ideal of near ring for the same. We show every strongly f -prime ideal is strongly prime but the converse is not true.

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1 Introduction

Throughout this article, N denotes a zero symmetric right near ring and $\mathbb{Z}_+$ denotes the set of positive integers. For any subset X of N , $\langle X \rangle$ denotes the smallest ideal containing X . For preliminary definitions and results related to near rings, we refer Pilz [3]. Let N be a near ring. A subset H of N is said to be an m -system if, for every $h_1, h_2 \in H$, there exists $h_1^1 \in \langle h_1 \rangle$ and $h_2^1 \in \langle h_2 \rangle$ such that $h_1^1 h_2^1 \in H$. A subset H of N is said to be an f -system if H contains an m -system H^* , called kernel of H , such that for

every $h \in H$, $f(h) \cap H^* \neq \phi$. An ideal A of N is said to be f -prime if $N \setminus A$ is an f -system. The f -prime radical of near ring N is defined as the intersection of all f -prime ideal of N and is denoted by $f\text{-rad}(N)$. An ideal $f(a)$ is a uniquely determined ideal by an element $a \in N$ which satisfies the following-: (i) $a \in f(a)$ (ii) $x \in f(a) + A \implies f(x) \subseteq f(a) + A$, for any ideal A of N . For any ideal I of a near ring N , we extend the map f on N/I to the set of all ideals of N/I such that $f(x + I) = (f(x) + I)/I, x \in N$. Then, this map satisfies the above mentioned two conditions (i) and (ii) for ideal $f(a)$.

An element $a \in N$ is said to be f -nilpotent if $f(a)$ is nilpotent. We introduced the concept of 2- f -primal near ring in [2]. A near ring N is said to be 2- f -primal near ring if $f\text{-rad}(N)$ equal to the set of all f -nilpotent element of N . An element $a \in N$ is said to be a strongly nilpotent if for every sequence $a_1, a_2, \dots$ in N satisfying $a_1 = a$ and $a_i = a_{i-1}^* a_{i-1}^{**}$, for some $a_{i-1}^*, a_{i-1}^{**} \in \langle a_{i-1} \rangle$, there exists an integer k such that $a_t = 0$, for $t \geq k$. An element a of N is said to be an f -strongly nilpotent if every element of $f(a)$ is strongly nilpotent. An ideal A of N is said to be strongly prime if A is prime and N/A has no non-zero nil ideals. Observe that every strongly prime is a prime ideal of N .

Lemma 1.1 $f\text{-rad}(N) = \{x \in N | x \text{ is an } f\text{-strongly nilpotent element}\}.$

Proof. See proof of Theorem 2.2 in [4] ■

Proposition 1.2 *Every f -strongly nilpotent element is an f -nil element, but converse is not true.*

Proof. Let x is an f -strongly nilpotent element. We show every element of $f(x)$ is nilpotent. If possible, let x is not nilpotent $\implies x^n \neq 0$, for any $n \in \mathbb{Z}_+$. Let $S = \{x, x^2, \dots\}$. Then S is an f -system with kernel S itself. By using Lemma [1.5 (a)] in [1], we have $(0) \subseteq P$ such that $P \cap S = \phi$. This implies, $x \notin P$ i.e. $x \notin \cap P = f\text{-rad}(N) = \text{set of all } f\text{-strongly nilpotent elements of } N$. This implies, x is not an f -strongly nilpotent element, which is a contradiction. Hence the result. ■

But the converse is not true.

Example 1.3 *Let T be an ideal of near ring N , which is nil but not nilpotent. Now, we define $f(a) = \langle a, T \rangle$, for all $a \in N$. Then, for any $t \in T$, $f(t) = T$ is not nilpotent. This implies, t is an f -nil but not f -strongly nilpotent.*

2 Strongly f -prime ideal

Definition 2.1 *An ideal I of the near ring N is said to be strongly f -prime if I is an f -prime ideal and N/I has no non-zero f -nil ideal. In another words, I is a strongly prime ideal if it is an f -prime and for any other ideal $J \supset I$, we have J/I is not an f -nil.*

Example 2.2 Consider the dihedral group $N = \{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$ with addition and multiplication given as in [scheme 84, [3]] defined as in the following table:

+	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	b	a+b	2a+b
b	b	3a+b	2a+b	a+b	0	3a	2a	a
a+b	a+b	b	3a+b	2a+b	a	0	3a	2a
2a+b	2a+b	a+b	b	3a+b	2a	a	0	3a
3a+b	3a+b	2a+b	a+b	b	3a	2a	a	0

.	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	b	2a+b	b	2a+b	b	b	b	2a+b
2a	0	0	0	0	0	0	0	0
3a	b	2a+b	b	2a+b	b	b	b	2a+b
b	b	b	b	b	b	b	b	b
a+b	0	2a	0	2a	0	0	0	2a
2a+b	b	b	b	b	b	b	b	b
3a+b	0	2a	0	2a	0	0	0	2a

There are 3 proper ideals of N , namely $A = \{0, 2a\}$, $A_1 = \{0, 2a, b, 2a+b\}$ and $A_2 = \{0, 2a, a+b, 3a+b\}$. Take $f(a) = \langle x, b \rangle$. Clearly, A is an f -prime ideal of N with kernel $A^* = \{b, 2a+b\}$. Also, $A_1/A = \{0, 2a, b, 2a+b\}$ is not an f -nil ideal, since b is not an f -nil element. Similarly, we can check that A_2/A is not an f -nil ideal. So, A is an f -strongly prime ideal of N .

Theorem 2.3 Let $M = \{\alpha, \alpha^2, \dots\}$, where α is not a nilpotent element of near ring N . Then there exists a strongly f -prime ideal P of N such that $P \cap M = \phi$.

Proof. Let $M = \{\alpha, \alpha^2, \dots\}$ and $S = \{I | I \text{ is an ideal of } N \text{ and } I \cap M = \phi\}$. Then $S \neq \phi$, since $(0) \in S$. Now, by using Zorn's Lemma, $\exists$ a maximal element say P , which turns out to be an f -prime ideal. Now, we show N/P does not have any non zero f -nil ideal. If possible, I/P is a non-zero f -nil ideal of N/P . Then, $P \subset I$ and so, $I \cap M \neq \phi$. This implies, any $x \in I = \alpha^m$, for some $m \in N$. Also, $x + P \in I/P$ is an f -nil element. This implies, $f(x+P) = (f(x)+P)/P$ is a nil ideal. Then, we can write $\alpha^{mn} \in P$, for some $m, n \in N$, which is a contradiction to the fact $P \cap M = \phi$.

■

Remark 2.4 Every strongly f -prime ideal is a strongly prime ideal. But converse is not true.

Example 2.5 From the Example [2.2], we have $A = \{0, 2a\}$ is a strongly f -prime ideal. But A is not a strongly prime, since A is not a prime ideal.

Proposition 2.6 In a near ring N , if every f -prime ideal is a strongly f -prime ideal. Then N is a 2- f -primal near ring.

Proof. For showing N is a 2- f -primal near ring, we prove $f\text{-rad}(N)$ contains all the f -nilpotent element of N . Let x is any f -nilpotent element of N . Then, $[f(x)]^n = (0)$, for some $n \geq 2$. If $x \notin f\text{-rad}(N)$. Then there exists an f -prime ideal P such that $x \notin P$. Using the assumption, P is a strongly f -prime ideal such that $x \notin P$. So, N/P has no non zero f -nil ideal and thus $((x) + P)/P$ is not an f -nil ideal. This implies, $x + P$ is not an f -nil. This implies, $(f(x + P))^n \neq P$, for any $n \in \mathbb{Z}_+$ and thus $(0) = [f(x)]^n$ is not contained in P , which is a contradiction. Hence the result. ■

Theorem 2.7 The intersection of a linearly ordered set of strongly f -prime ideals of near ring N is a strongly f -prime ideal.

Proof. Let $\{P_k | k \in \Lambda\}$ be a family of linearly ordered strongly f -prime ideals of near ring N and $P = \cap P_k, k \in \Lambda$. Then each P_k , being an f -prime ideal has its complement $C(P_k)$ as an f -system with an m -system P_k^* (say). First, we show P is an f -prime ideal. Consider $P^* = \cup P_k^*$, union is taken over Λ . Clearly, P^* is an m -system, since for any $\alpha, \beta \in P^*$, there exist k_1, k_2 such that $\alpha \in P_{k_1}^*$, and $\beta \in P_{k_2}^*$. Then, by using Proposition 2.92 of [3], we have $\alpha, \beta \in P_{k_1}^*$ (say). Since each $P_{k_1}^*$ is an m -system. So, $(\alpha)(\beta) \cap P_{k_1}^* \neq \phi$. This implies, $(\alpha)(\beta) \cap P^* \neq \phi$. Now, $P^* = \cup P_k^* \subseteq C(\cap P_k) = C(P)$. Let $a \in P \Rightarrow a \in P_k$, for some k . So, there exists $(P_k)^* \subseteq C(P_k)$ such that $f(a) \cap P_k^* \neq \phi$. This implies, $f(a) \cap P^* \neq \phi$, for all $a \in P$.

Also, we show N/P does not have any non zero f -nil ideal, when N/P_k does not have any non-zero f -nil ideal, for any k . Let $J/P, P \subset J$ is a non zero f -nil ideal of N/P . Now, $P_k \subset P = \cup P_k \subset J$. So, J/P_k is a non-zero f -nil ideal of N/P , for any k , which is a contradiction. Hence the result. ■

Definition 2.8 An element $a \in N$ is said to be f -nil if $f(a)$ is nil ideal.

Definition 2.9 A subset H of a near ring N is said to be f -nil if every element of H is f -nil.

Lemma 2.10 If A be an f -nil ideal of a near ring N and M/A be an f -nil subset of quotient near ring N/A . Then M is f -nil.

Proof. Let $m \in M$ be any element. We show $f(m)$ is nil ideal. Since, for any $m \in M, m + A \in M/A$ is an f -nil element in N/A . So, $f(m + A) = (f(m) + A)/A$ is a nil ideal in N/A . This implies, $\hat{m} + A \in (f(m) + A)/A$ is nilpotent in N/A and thus $(\hat{m} + A)^n = A$, for some $n \in \mathbb{Z}_+ \Rightarrow \hat{m}^n \in A$, which is nil, since A is f -nil. $\Rightarrow \hat{m}$ is nilpotent. So, $f(m)$ is nilpotent, for any $m \in M$. Thus M is f -nil ideal. Hence the result. ■

Theorem 2.11 *Any near ring with unity has a unique maximal f -nil ideal, if it exists.*

Proof. The proof follows by using Zorn's Lemma on the collection of all f -nil ideal of near ring N . So, it is required to show the maximal f -nil ideal say M in near ring N is unique. For this, it is enough to show that M contains every other f -nil ideal I . It is clear that $(M + I)/M$ is an f -nil subset of N/M . Then by using Lemma [2.10], $M + I$ is an f -nil. But M being a maximal ideal, implies $M + I = M \Rightarrow I \subseteq M$. Hence the result. ■

Theorem 2.12 *In any near ring N , Intersection of all strongly f -prime ideal of N is equal to the unique maximal f -nil ideal $\eta_f(N)$.*

Proof. Let $\eta_f(N)$ is not contained in P , for some strongly f -prime ideal of N . Then $\eta_f(N)/P$, is a non-zero f -nil ideal of N/P . But P , being a strongly f -prime ideal of near ring N , N/P cannot have a non zero nil ideal, which is a contradiction. Hence, $\eta_f(N) \subseteq$ intersection of all strongly f -prime ideal of N . Conversely, we show the intersection of all strongly f -prime ideal of N is an f -nil ideal and so is equal to $\eta_f(N)$. Let $x \in N$ is not an f -nil element. Then $f(x)$ is not a nil ideal. This implies, $f(x)$ cannot be an nilpotent ideal of N . Then there exists a strongly f -prime ideal which does not contain $f(x)$. Let $S = \{f(x), [f(x)]^2, \dots\}$. Then by using Zorn's Lemma, $\exists$ some ideal P maximal with respect to $P \cap S = \phi$, which is an f -prime ideal. If N/P has a non-zero f -nil ideal say M/P , then $P \subset M$ and so, $M \cap S \neq \phi \Rightarrow \alpha^n \in M$, where $\alpha = f(a)$ and $n \in \mathbb{Z}_+ \Rightarrow \alpha^n \in P$, which is a contradiction. Hence the result. ■

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