Mathematical Model for the Deterministic Extended Machine Replacement Policy

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Abstract

This paper uses iterative method for replacement and maintenance policies leading to a very efficient model in large systems unlike the existing exhaustive enumeration algorithm which can be used only if the number of stationary policies is reasonably small. From the numerically simulated results, it was observed that the optimal values of unbounded horizon problems are obtained at the last peak of the model before shooting into non convergence state. Immediately after the optimum report is acquired, a local minimum is achieved. Thereafter, a non convergence positive result that goes into infinity is achieved. The last trough clearly gives a false positive respond. Therefore, the optimum value observed before non-convergence state should be the decisive state to quit the stage. All equipment must be constantly kept under good maintenance since the cost of maintenance increases with time. At a stage of deterioration, equipment could be refurbished and sold out at less expensive cost as their life-span cannot stand the test of time.

Keywords: Deterministic model, unbounded horizon, maintenance, machine and replacement

Introduction

Cho and Parlar (1991) defined maintenance as a combination of actions carried out to restore the structure's component or to "renew" it to the original condition. Inspections, repairs, replacements, and lifetime-extending measures are possible maintenance actions. Through lifetime-extending measures, the deterioration can be delayed such that failure is postponed and the component's lifetime is extended. Maintenance may be categorized into two, these are: *corrective maintenance* (mainly after failure) and *preventive maintenance* (mainly before failure). Corrective maintenance can best be chosen if the cost arising from the failure is low (such as replacing a burnt-out light bulb); preventive maintenance if this cost is high (like for heightening a dyke). In structuring engineering, for instance, the consequences of failure are generally so large that mainly expensive preventive maintenance is applied. The use of maintenance optimization models is therefore of considerable interest Hakimi, (2003).

Mode and Pickens (1998) identified four phases of a lifetime structure: the design, the building, the use, and the demolition. There are mainly two phases in which it is worth applying maintenance optimization techniques: (i) the design phase and (ii) the use phase. In the design phase, the initial cost of investment has to be balanced against the future cost of maintenance. In the use phase, the costs of inspection and preventive replacement have to be balanced against the cost of corrective replacement and failure Hakimi (2011).

Review of Literatures

Blackwell (2002) opined that the notion of equivalent average costs relates to the notions of average costs and discounted costs. The cost-based criteria of discounted costs and equivalent average costs are most suitable for balancing the initial building cost optimally against the future maintenance cost. The criterion of average costs can be used in situations in which no large investments are made (like inspections) and in which the time value of money is of no consequence to us. Often, it is preferable to spread the costs of maintenance over time and use discounting (Denardo and Miller, 2008).

Examples of optimizing maintenance in the design phase are: determining optimal dyke heightening and optimal sand nourishments who's expected discounted costs are minimal (Tersine, 1998 and Noortwijk & Peerbolte, 2000). Examples of optimizing maintenance in the use phase are: determining cost-optimal rates of

inspection for dykes, berm breakwaters, and the sea-bed protection of the Eastern-Scheldt barrier (Noortwijk *et al.* (1995, 1999, 1997, 1996)), and determining cost-optimal preventive maintenance intervals. Maintenance of structures can often be modeled as a discrete renewal process, whereby the renewals are the maintenance actions that bring a component back into its original condition or “as good as new state”. After each renewal, it is started, in statistical sense, all over again.

Elkins and Wortman (2002) defined discrete renewal process $\{N(n): n \in \mathbb{N}\}$ as a non-negative integer-valued stochastic process that registers the successive renewals in the time-interval $(0, n]$. Let the renewal times $T_1, T_2, \dots$, be non-negative, independent, identically distributed, random quantities having the discrete probability function $\Pr\{T_k = i\} = p_i(d)$, $i \in \mathbb{N}$, with $\sum_{i=1}^{\infty} p_i(d) = 1$, where $p_i(d)$ represents the probability of a renewal in unit time i when the decision-maker chooses maintenance decision d . We denote the costs associated with a renewal in unit time i by $c_i(d)$, $i \in \mathbb{N}$. The above-mentioned three cost-based criteria will be discussed in more detail in the following subsections after our full analysis of the finite horizon in the limit.

In a characteristic dexterity as expunged by Wagner (1989, 2003), we now treat recursive approximation problems:

Materials and Methods

We derived the deterministic replacement model. In this policy, a machine is considered as a system that consists of a single machine and the machine is assumed to operate continuously. The downtime of the machine for repair is assumed to be negligible. Every new machine for replacement is conceived to be identical. In such a model as this one, we consider the maintenance and the replacement costs. The cost rate of maintaining a machine of age t is $p(t)$ and the replacement cost is K . In this analysis, $p(t)$, is conceived to be smooth (continuously differentiable) and is strictly increasing with respect to t . Therefore, it is natural to have a replacement policy for the machine when the planning is an unbounded horizon problem.

The objective here is to minimize the average long-run cost due to the replacement and maintenance in the system. We note that whenever an old machine is replaced, a new cycle starts (the renewal stage). Since all the machines are assumed to be identical, all the virtual cycles are the same. In this case, one needs only to consider the fact that all the cycles are the same. Therefore, one

needs only to consider the average running cost in a cycle. The replacement cost per cycle (a period of time t) is of course K and the maintenance cost per cycle is given by

$$M(t) = \int_0^t p(t)dt \Rightarrow p(t) = M'(t) \quad (1)$$

Therefore the average cost is

$$C(t) = \frac{K + M(t)}{t} = \frac{K}{t} + \frac{M(t)}{t} \quad (2)$$

With this, it is not difficult to establish the following proposition

If $p(t)$ is strictly convex, that is $p''(t) > 0$, then the optimal value of t is the zero of the equation

$$K + \int_0^t p(t)dt - tp(t) = 0 \quad (3)$$

We note that for the rate of average costs $C(t)$, the first and second derivatives can be obtained in the following sequential analytical order:

Now,

$$C(t) = \frac{K}{t} + \frac{M(t)}{t}$$

$$\therefore C'(t) = \frac{-K}{t^2} - \frac{M(t)}{t^2} + \frac{M'(t)}{t} = \frac{tM'(t) - M(t) - K}{t^2} = \frac{tP(t) - M(t) - K}{t^2} \quad (4)$$

and

$$C''(t) = \frac{t^2(P(t) + tP'(t) - M'(t)) - 2t(tP(t) - M(t) - K)}{t^3} \quad (5)$$

$$= \frac{tP(t) + t^2P'(t) - tP(t) - 2tP(t) + 2M(t) + 2K}{t^3}$$

$$= \frac{t^2P'(t) + 2M(t) + 2K - 2tP(t)}{t^3}$$

Let

$$H(t) = t^2 P'(t) + 2M(t) + 2K - 2tP(t) \quad (6)$$

Then,

$$C''(t) = \frac{H(t)}{t^3}$$

and therefore;

$$\begin{aligned} H'(t) &= 2tP'(t) + t^2 P''(t) + 2M'(t) - 2P(t) - 2tP'(t) \\ &= t^2 P''(t) + 2M'(t) - 2M'(t) \\ \therefore H'(t) &= t^2 P''(t) \end{aligned} \quad (7)$$

Now, since $p''(t) > 0$, $H(t)$ is an increasing function in t , hence $C''(t) > 0 \forall t > 0$.
Next we define

$$R(t) = K + \int_0^t p(t)dt - tp(t) \quad (8)$$

Since

$$R'(t) = p(t) - p(t) - tp'(t) < 0 \text{ for } t > 0 \quad (9)$$

$R(t)$ is strictly decreasing for $t > 0$. Moreover, we have

$$\int_0^t p(t)dt - tp(t) = \int_0^t [p(t) - p(t) - tp'(t)]dt = -\int_0^t tp'(t)dt \quad (10)$$

Therefore we deduce intuitively that

$$R(0) = K > 0 \text{ and } \lim_{t \rightarrow \infty} R(t) = -\infty$$

This means that $C(t)$ has a unique minimum in $(0, \infty)$ and from, $C'(t) = 0$, it is the root of equation

$$K + \int_0^t p(t)dt - tp(t) = 0 \quad (11)$$

The root can be obtained by using bisection method. This equation (11) is **our new Extended Machine Replacement model**. Once the cost of maintenance equals the cost of the new machine, the machine ought to be replaced with a new one.

Now suppose that the current cost rate of maintaining the machine is defined as

$$p(t) = \alpha t^\beta \quad (12)$$

where by definition $\beta > 2$ and $\alpha > 0$, then $p''(t) > 0$ and

$$C'(t) = \frac{1}{t^2} (-K + \alpha t^{\beta+1} - \int_0^t \alpha t^\beta dt) = -Kt^{-2} + \frac{\alpha\beta}{\beta+1} t^{\beta-1} \quad (13)$$

The optimal planned replacement period is $\sqrt[\beta+1]{\frac{(\beta+1)K}{\alpha\beta}}$.

Illustration

The purchase value of a machine is #10, 000. 00. The running/ maintenance cost is estimated at #1, 000.00 per annum for the first five-years, increasing by #300 in the sixth and the subsequent years. If the prevailing rate of interest is 10 percent per annum: (1) When is the optimum time (n) to replace the machine? (2) What amount of money should be set aside each year for the replacement of the machine with a new one?

With the application of our model, table 1 gives the solution. It shows that the machine should be replaced at the end of 11 years:

Table 1: Solution to the Replacement machine:

<i>Year</i> (<i>t</i>)	<i>C</i> Initial cost	<i>R_t</i> Mainte nance cost	$1-V$ Discou nt cost (consta nt	$1-V^n$ nth cost factor	V^{t-1} (t-1) period cost factor	$C + V^{t-1}R_t$ present maintenan ce cost	$P_n = C \sum V^{t-1} \frac{(R_t V^{n-1})}{(1-V)} R_{t+1}$ present Total (cumulative) cost value	Present cumulative maintenance cost value
1	10000	1000	0.0909	0.0909	1.0000	11000.00	11000.00	1000.00
2	0	1000	0.0909	0.1736	0.9091	909.10	11909.10	1909.79
3	0	1000	0.0909	0.2487	0.8264	826.40	12375.50	2735.97
4	0	1000	0.0909	0.3170	0.7513	751.30	13486.80	3487.35
5	0	1000	0.0909	0.3791	0.6830	683.00	14169.80	5421.67
6	0	1300	0.0909	0.4355	0.6209	807.17	14976.97	7665.57
7	0	1600	0.0909	0.4868	0.5645	903.20	15880.17	10175.14
8	0	1900	0.0909	0.5335	0.5132	975.08	16855.25	12911.99
9	0	2200	0.0909	0.5759	0.4665	1026.30	17881.55	15838.83
10	0	2500	0.0909	0.6145	0.4241	1060.25	18941.80	18928.49
11*	0	2800	0.0909	0.6495	0.3855	1079.40	20021.20	22150.17
12	0	3100	0.0909	0.6814	0.3505	1086.55	21107.75	25486.91
13	0	3400	0.0909	0.7103	0.3186	1083.24	22190.99

Figure 1: Curve Fitting Analysis to the illustration

Computational analysis of figure 1 is analyzed numerically below:

Fit computation did not converge:

Maximum number of function evaluations exceeded. Increasing MaxFunEvals (in fit options) may allow for a better fit, or the current equation may not be a good model for the data.

Fit found when optimization is terminated:

fittedmodel3 =

General model Gauss4:

$fittedmodel3(x)$

$$= -9017 * e^{-\left(\frac{x-11.72}{0.5306}\right)^2} + 4.668 * 10^{12} * e^{-\left(\frac{x-85.77}{15.96}\right)^2} + 11540 * e^{-\left(\frac{x-9.345}{4.128}\right)^2} + 1441 * e^{-\left(\frac{x-3.501}{3.695}\right)^2}$$

Goodness of fit:
SSE: 1.522e+005
R-square: 0.9998
Adjusted R-square: NaN
RMSE: NaN

The curves depict non-convergence structures as the derivatives of both the 1st and 2nd derivatives show convexities.

Discussion of Results

Unbounded horizon problems are characterized as shown in the numerical analysis and graphical exhibitions of the illustration. First, the optimal values of unbounded horizon problems are obtained at the last peak of the model before shooting into non convergence state. This assertion is clearly buttressed from the graphical inclinations in figure 1. Second, immediately after the optimum report is acquired, a local minimum is achieved. Thereafter, a non convergence positive result that goes into infinity is achieved. The last trough clearly gives a false positive respond. Therefore, the optimum value observed before non-convergence state should be the decisive state to quit the stage.

Conclusion

In conclusion, unbounded horizon problems do not converge to a fixed optimum. In view of this, for any maintenance or replacement policy in terms of cost benefits, the decision maker must stop any unreasonable expenses on the maintenance of such a venture or he will be at a losing state. The optimum stage is the state in which replacement of any maintaining policy can be stopped. Any further incurrence of maintenance cost would result to loss in the future. The general frame of chapter two on the Literature dwells mostly on making optimal decisions based on Maintenance policies in such major areas as replacement and renewal policies. The structures of the computations were mostly in tabular forms.

These iterative numerical formats exhibited a high degree of computations as the recursive optimal values were numerically visualized which aided the determination of optimal decisions.

Recommendations

The following recommendations are made:

- (i) All equipment must be constantly kept under good maintenance. Under this observation, the cost of maintenance increases with time. Therefore, it is bound to reach a peak so that we need to forsake the use of such services of the equipment as the cost of its maintenance becomes expensively prohibitive.
- (ii) At such stage of deterioration, equipment could be refurbished and sold out at less expensive cost as their life-span cannot stand the test of time.
- (iii) The model is very useful for maintenance policies within governmental and institutions alike.

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