Development of Iterative Method for Replacement and Maintenance Process Using Inventory Model

D. Hakimi, A. U. Yusuf, K. R. Adeboye

Department of Mathematics and Statistics
School of Applied Natural and Applied Sciences
Federal University of Technology
Minna, Niger State, Nigeria

V. O. Waziri

Cyber Security Science Department
School of Information and Communication Technology
Federal University of Technology
Minna, Niger State, Nigeria

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Abstract

In this paper, we developed new model on replacement policy for some machines in some economic setup using discounted factors and Markov chain processes. Computational processes were applied to solve some proposed unbounded optimization problem which included iterative method for replacement and maintenance policies using inventory model. The conventional inventory model only balanced off manufacturing with inventory holding costs while the economic cost of varying production levels from one period to the next was ignored and equally ignored discount factor knowing fully that money depreciates with time.

All these deficiencies are taken cared by our newly developed inventory model. The new model is also very efficient even in large systems unlike the existing exhaustive enumeration algorithm which can be used only if the number of stationary policies is reasonably small. From the numerically simulated results, it was observed that the optimal values of unbounded horizon problems were

obtained at the last peak of the model before shooting into non-convergence state. It was also observed that after the optimum report was acquired, a local minimum was achieved and thereafter, a non-convergence positive result that went into infinity followed. Manufacturing industries can apply the inventory model developed in the acquisition of raw materials as the stocks are replenished at the right time.

Keywords: Replacement policy, unbounded optimization, iterative method, non-convergence state, stocks and raw materials

Introduction

Mathematicians and economists have for many hundreds of years assess methods for acquiring models that can give effective returns for infinite streams of returns or unbounded stream of return (Hakimi, 2011). This paper investigates two questions that are central to optimization in dynamic settings:

- (i) When is an evaluation model appropriate for comparing different strategies?
- (ii) Does such a model always reduce an infinite stream to a single number that can be used as the basis for comparison?

We respond to these two points in greater details in the subsequent models below. We investigated three models of merit. The models are (a) the average return per period, (b) the present discounted value and (c) equivalent average return. We developed optimization model for Infinite Decision-making and Application which will involve the use of Inventory Model to a practical problem. Manufacturing industries can apply this inventory model in the acquisition of raw materials as the stocks are replenished at the right time.

Literature Review

The expected lifetime of a physical structure is usually long and as such; maintenance decisions may be compared over an unbounded horizon. Wagner (1989, 2003) considered basically three cost-based criteria that can be used to compare maintenance decision. These are:

1. The expected average costs per unit time, which is determined by averaging the costs over an unbounded horizon;
2. The expected discounted costs over an unbounded horizon, which are determined by summing the (present) discounted values of the costs over an unbounded horizon, under the assumption that the value of money decreases with time;

3. The expected equivalent average costs per unit time, which is determined by calculating the discounted costs per unit time.

Cho *et al.* (1991) defined maintenance as a combination of actions carried out to restore the structure's component or to "renew" it to the original condition. Inspections, repairs, replacements, and lifetime-extending measures are possible maintenance actions. Through lifetime-extending measures, the deterioration can be delayed such that failure is postponed and the component's lifetime is extended. Maintenance may be categorized into two, these are: corrective maintenance (mainly after failure) and preventive maintenance (mainly before failure). Corrective maintenance can best be chosen if the cost arising from the failure is low (such as replacing a burnt-out light bulb); preventive maintenance if this cost is high (like for heightening a dyke). In structuring engineering, for instance, the consequences of failure are generally so large that mainly expensive preventive maintenance is applied. The use of maintenance optimization models is therefore of considerable interest.

Mode *et al.* (1998) identified four phases of a lifetime structure: the design, the building, the use, and the demolition. There are mainly two phases in which it is worth applying maintenance optimization techniques: (i) the design phase and (ii) the use phase. In the design phase, the initial cost of investment has to be balanced against the future cost of maintenance. In the use phase, the costs of inspection and preventive replacement have to be balanced against the cost of corrective replacement and failure.

The notion of equivalent average costs relates to the notions of average costs and discounted costs. The cost-based criteria of discounted costs and equivalent average costs are most suitable for balancing the initial building cost optimally against the future maintenance cost. The criterion of average costs can be used in situations in which no large investments are made (like inspections) and in which the time value of money is of no consequence to us. Often, it is preferable to spread the costs of maintenance over time and use discounting (Blackwell, 2002; and Denardo and Miller, 2008).

Examples of optimizing maintenance in the design phase are: determining optimal dyke heightening and optimal sand nourishments who's expected discounted costs are minimal (Tersine, 1998; and Noortwijk and Peerbolta, 2000). Examples of optimizing maintenance in the use phase are: determining cost-optimal rates of inspection for dykes, berm breakwaters, and the sea-bed protection of the Eastern-Scheldt barrier and determining cost-optimal preventive maintenance intervals.

Maintenance of structures can often be modeled as a discrete renewal process, whereby the renewals are the maintenance actions that bring a component back into its original condition or "as good as new state". After each renewal, it is started, in statistical sense, all over again.

Elkins and Wortman (2002) defined discrete renewal process $\{N(n): n \in \mathbb{IN}\}$ as a non-negative integer-valued stochastic process that registers the successive renewals in the time-interval $(0, n]$. Let the renewal times $T_1, T_2, \dots$, be non-negative, independent, identically distributed, random quantities having the discrete probability function $\Pr\{T_k = i\} = \pi_i(d)$, $i \in \mathbb{IN}$, with $\sum_{i=1}^{\infty} \pi_i(d) = 1$, where $\pi_i(d)$ represents the probability of a renewal in unit time i when the decision-maker chooses maintenance decision d . We denote the costs associated with a renewal in unit time i by $c_i(d)$, $i \in \mathbb{IN}$. The above-mentioned three cost-based criteria will be discussed in more detail in the following subsections after our full analysis of the finite horizon in the limit.

Hakimi (2003) and Brown (2005) initiated the numerical techniques for solving extremal equations which arise in dynamic models having an unbounded horizon. The prototype formula is the functional equation defined as:

$$f_n = \min_{k=1,2,\dots,N} [\alpha^k f + R_k] \quad f_0 = 0 \text{ for } 0 \leq \alpha < 1 \quad (1)$$

where by definition, equation (1) symbolizes the present optimal regeneration policy in which an alternative within the maintenance period must be chosen when n period remains until the end of the planning horizon. Suppose that alternative k is chosen, then we will immediately incur the cost R_k and should we act optimally in our choice at the next generative period of maintenance, $n-k$, then the cost for the present maintenance will be $\alpha^k f_{n-k}$, where the factor α^k significantly discounts the future cost to the present. Thus the optimal choice when n periods remaining until the end of the horizon is a policy that minimizes the sum $\alpha^k f_{n-k} + R_k$ and the corresponding minimum value is f_n . Hence with $n \geq N$, f_n can be characterized recursively by the relation

$$f_n = \min_{k=1,2,\dots,N} [\alpha^k f_{n-k} + R_k] \quad f_0 = 0, \text{ for } 0 \leq \alpha < 1 \quad (2)$$

In a recursive representation, suppose $k=1$ is optimal for all horizon length, then equation (2) is expressible as:

$$\begin{aligned} f_n &= \alpha f_{n-1} + R_1 = \alpha[\alpha f_{n-2} + R_1] + R_1 \\ &= \alpha[\alpha(\alpha f_{n-3} + R_1) + R_1] + R_1 \\ &= \dots = R_1 + \alpha R_1 + \dots + \alpha^{n-1} R_1 \end{aligned} \quad (3)$$

Equation (3) is a useful model for the computation of finite horizon bounded combinatorial optimization problems. The generality of this model is extensible into the unbounded horizon.

Hopkins (1997) and Harrison (2002) considered the situation where the planning for the replacement of the equipment is regenerative process. Each time the regeneration occurs, the decision-maker continues to experience unlimited horizon. In an unbounded horizon, if there exists an optimal strategy (or policy) that is stationary. Thus if $\alpha \neq 1$, the appropriate generalization of equation, the appropriate generalization of (2) is definable as

$$f = \min_{k=1,2,\dots,N} (\alpha^k f + R_k) \text{ for } 0 \leq \alpha < 1 \quad (4)$$

Unlike the previous analysis in the introductory part of chapter two, the $P.V.[k]$, being the present value, it does depend on the discounted value α . Also under this generative unbounded horizon, k is assumed a priori to be N . Equation (4) is an example of what is called a functional or extremal equation. It is the value of f that is unknown, and it states the optimization relation which f must satisfy, given that a stationary model is used. In such situation, when dealing with extremal equations, we must always determine:

- (i) Does the equation possess a finite solution?
- (ii) If so, is the solution unique?
- (iii) If so, is f the maximal discounted return among all (not necessarily stationary) policies?

To see the relevance of equation (4) suppose $\alpha = 1$ (Dirickx, 2003), contrary to restriction on the right of equation (4). If we assume that all $R_k > 0$, then no finite value for f satisfies (4). Therefore, the functional equation (4) is not appropriate for, $\alpha = 1$. Thus, we can view (4) as

$$f \leq \alpha^k f + R_k \quad \text{or} \quad f \leq \frac{R_n}{1-\alpha^k} \quad \forall k, \quad (5)$$

and equality in (5) must hold for at least one value of k . it follows that a unique finite solution to the external equation (4) does exist and equals

$$f = \min_{k=1,2,3,\dots,N} \left[\frac{R_k}{1-\alpha^k} \right] \quad (6)$$

An optimal stationary policy corresponds to any Alternative k that yields the optimal value for f . The derivation of (6) is also possible on the basis of stationarity. Since it is optimal to employ the same Alternative k every time a regression occurs, the present value of the policy is

$$R_k + \alpha^k R_k + \alpha^{2k} R_k + \alpha^{3k} R_k + \alpha^{4k} R_k + \dots = \frac{R_k}{1-\alpha^k} \quad (7)$$

Thus an optimal policy is the one that minimizes this quantity, as indicated in equation (6). Hence infinite has been solved using $\alpha \neq 1$.

Method and Material

This subsection initiates the numerical technique for solving extremal equations that arise in dynamic programming models having an unbounded horizon. Using the prototype functional model from equation (4), Winston (1994) and Cani (2004) said there are three (3) models that can be exploited based on ideas.

(1) The first model estimates from the dynamic context of the underlying model. The principal conception is to visualize whether a policy which is optimal for a very long, but finite horizon yields a solution value for f , when used over unbounded horizon.

(2) The second idea is to guess a value for f . Then compute the quantity on the right-hand side of equation (4) using the guess value to whether the equation is satisfied. Otherwise, let the result of the computation be the revised guess and then repeat the process.

(3) The third model is to guess a policy that may be optimal over an unbounded horizon. Then solve for the corresponding present value, and use it as a trial value for f to see if the (4) is satisfied; otherwise, let the new guess value be the policy which gives minimum on the right-hand side of (4) and repeat the process ad-infinitum until optimal policy attainment is accomplished.

In these guess methods, each guess can be viewed as an approximation to the solution, in other words, the solutions are derived from heuristic processes. If the guess satisfied the extremal equation, it is done. If not, one must guess again. The iterative process is given the label successive approximations. We now consider the three models within sequential contexts:

Derman and Klein (2005); and Karp and Held (2007) identify some obvious approaches for finding a policy which yields a solution to the functional equation (4). It is to solve the finite horizon model

$$f_n = \min_{k=1,2,\dots,N} [\alpha^k f_{n-k} + R_k] \quad f_0 = 0, \text{ for } 0 \leq \alpha < 1 \quad (8)$$

for every large value of n . It is significant that for the regeneration model there exists a finite value n^* such that for any finite horizon $n > n^*$, if

$$f_n = \alpha^{k_n} f_{n-k_n} + R_{k_n}, \text{ then } f = \alpha^{k_n} f + R_{k_n} \quad (9)$$

$$f = \alpha^K f + R_K, \text{ then } f = \alpha^K f_{n-K} + R_K \quad (10)$$

Equation (8) stipulates that any strategy k_n which is optimal for the current decision when the horizon n is large [greater than n^*] is also an optimal stationary strategy for an unbounded horizon. Model (9) asserts the reverse proposition. By performing the calculations on model (1) according to a certain computation format, one can ascertain n^* . The details of the approach are extraneous of the purpose of this discussion, and therefore are omitted.

This subsection is to successively approximate the function f , in the extremal equation. Hence the idea is the process termed value iteration Denardo (1992). Suppose we let f^0 be the initial approximation, then the technique is to compose a sequence of approximations $f^0, f^1, f^2, \dots$, according to the recursion

$$f^{n+1} = \min_{i=1,2,3,\dots,N} [\alpha^i f^n + R_i] \text{ for } 0 \leq \alpha < 1 \quad (11)$$

where f^n is the trial value for f from iteration n . If the approximation in the extremal equation indicates “maximization”, then the corresponding change is made in equation (1). Here, we give an example of the method. Although the algorithm is well positioned, three questions arise about its applications:

- (i) Does the value of f^n always approach the value of f that satisfies the extremal equation?
- (ii) If so, is there a finite n such that f^n equals f ?

(iii) If alternative k is chosen in equation (1) for two successive approximations is it optimal?

Suppose for the moment that $R_n > 0$. Let $f^0 = 0$, then it can be proved that $f^{n+1} > f^n$, so that a monotonically increasing sequence of approximations. And for n sufficiently large, f^n is such that f^n equals f . In general, there is no finite n such that f^n equals f , and further, an alternative may be chosen on the right-hand side of equation (1) for the two or more successive approximations but need not be optimal in an unbounded horizon.

The recursive term f^{n+1} can be taken as the present value of repeatedly chosen policy. This recursive procedure is the policy iteration since each iteration considers a new trial stationary policy for the unbounded horizon. The resultant computational sequence is monotonically decreasing, and it is their improvement which occurs at every iteration, this ensures that one never returns to a policy once it has been discarded Dreyfus (1997).

The algorithm is:

Step1. Select an arbitrary initial policy, and let $n=0$.

Step2. Given the trial policy, calculate the associated

$$f^n = \frac{R_k}{1 - \alpha^k} \text{ (present value of trial } k \text{ over an unbounded horizon)} \quad (12)$$

Step3. Test for an improvement by calculating

$$\min_{1,2,3,\dots,N} [\alpha^k f^n + R_n] = \alpha^{k'} f^n + R_{k'} \text{ (select } k') \quad (13)$$

Step4. Terminate the iterations if $\alpha^{k'} f^n + R_{k'} = f^n$. Otherwise, revise the policy in k' . Increase n to $n+1$, and return to step 2 with the new trial policy.

We consider a particular computational derivation based on replacement optimization problem. In a characteristic dexterity as in (Chan *et al.*, 2005 and Kolesar, 2006), we considered replacement problems in which replacement problem is observed as optimization problem involving cost minimization or profit maximization as the criterion function. The treatment of both models has similar conception as the unbounded horizon analysis considered so far.

In a characteristic dexterity as in Klein (2002), Ralymon (2002) and Derman (2007), we considered heavy-duty equipment such as cars, trailers, planes and

motor cycles to have their services exponentially distributed over some epoch of time. Besides the initial cost of the purchase, other contemporaneous costs are sustained annually for running, maintenance and repairs. These concomitant costs tend to rise as the equipment gets older. Hence, it is essential over this pressure to know the most despicable moment to replace the existing equipment with some new one. This essentially is a replacement fundamental problem.

Ching (1998) and Ching *et al.* (2004 and 2005) deduced intuitively that the replacement problem may be conceived or seen as one of knowing when the running/ maintenance cost becomes so high that the discounted total value is higher than the cost of buying a new machine. Now let us define their theoretical model:

- (1) An entrepreneur would not have bought equipment in the first place if it would not pay him to buy it.
- (2) He would not want to replace the machine if it would not pay to replace it.

Assumptions (1) and (2) imply that the entrepreneur is rational and the expectation of profitability in the use of equipment is assumed.

- (1) Once the equipment is brought into use, continued profitability of use is assumed.
- (2) An equipment when due for replacement, would be replaced with identical one.
- (3) The equipment is required for an indefinite number of future periods; that is an equipment is required over an unbounded horizon.

We also considered the following parameters:

P_n = Total (cumulative) present value of all future costs a period
over a period of n years

R_t = Running / maintenance cost of equipment in period t .

r = The discount rate

n = Optimum replacement period which is to be determined

C = initial purchase price of equipment

With the defined parameters, we obtained the model:

$$P_n = C + R_1 + \frac{R_2}{(1+r)} + \frac{R_3}{(1+r)^2} + \frac{R_4}{(1+r)^3} + \dots + \frac{R_n}{(1+r)^{n-1}} \quad (14)$$

Thus the present value of all future costs a period is identical to equivalent discounted value such as (14) is defined as

$$P_n = C + R_1 + \alpha R_2 + \alpha^2 R_3 + \dots + \alpha^{n-1} R_n \quad (15)$$

Where $\alpha = (1 + \frac{i\%}{100})^{-1}$ is the discounted factor.

Equations (15) may be written compactly as

$$P_n = C + \sum_{i=1}^n \alpha^{i-1} R_i \quad (16)$$

That is all cost are discounted to the initial horizon $t=1$

P_n is the anticipated amount that would be needed to replace the equipment after some period of unforeseeable future. In other words, it could be seen as the total amount that is needed to purchase and maintain the current piece of equipment over an unbounded horizon. In optimization subtext, P_n is the optimal amount that is needed to replace the equipment over an unbounded horizon (Jewell, 2003). Obviously, P_n may be viewed as the amount of money on hand at the end of the optimal unbounded horizon or n years when new equipment shall be ready for procurement or replacement. This computational present amount is short circuited if we can idealize a fixed nominal amount of money which in financial parlance is known as annuity or discount value. This value may be contrived in sequential order as “ α_t ” naira which is set aside each year, will be exactly equal, in discounted value, to P_n . That is, we need

$$\sum_{i=1}^n \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = P_n \quad (17)$$

The model is derived in a clear exposition:

$$\alpha_1 = \alpha_1, \alpha_2 = \frac{\alpha_1}{1+r}, \alpha_3 = \frac{\alpha_1}{(1+r)^2}, \dots, \alpha_n = \frac{\alpha_1}{(1+r)^{n-1}} \quad (18)$$

Setting $\frac{1}{1+r} = v$, we can express (17) explicitly as

$$\sum_{i=1}^n \alpha_i = \alpha_1 + \alpha_1 V + \alpha_1 V^2 + \alpha_1 V^3 + \alpha_1 V^4 + \dots + \alpha_1 V^{n-1} = P_n \quad (19)$$

This can be simplified to;

$$\sum \alpha_t = \frac{\alpha_1(1-V^n)}{1-V} \quad (20)$$

from which the amortized present value can be evaluated.

From equation (20), we can deduce that α_1 is the nominal amount of money which can be saved every year so that, given an annual discount rate “r”, the accumulated sum after n years will just be enough to purchase a new equipment and run the new equipment for a subsequent period of n years.

Now comparing equations (17) and (20), we obtained that

$$\alpha_1 = P_n \frac{(1-V)}{(1-V^n)} \quad (21)$$

The next step is to compute the minimum annuity that can be set aside over the unbounded horizon such that its accumulative present value would be equal to P_n . This clearly is the problem of minimizing positive with respect to n years. Logically, since $(1-V)$ in equation (21) is a constant, whenever (21) is minimized,

it could be conspicuously seen that $\frac{P_n}{(1-V^n)}$

Hence, it is possible to minimize equation (21) to the optimal function such that it is expressible as:

$$F(n) = \frac{P_n}{(1-V^n)} \quad (22)$$

with respect to the unbounded period n.

Now that n is a discrete variable, the first-order condition for a minimum can be attained by the method of finite differences.

$F(n)$ is minimized when

$$\Delta F(n-1) = F(n) - F(n-1) < 0 \quad (23)$$

and

$$\Delta F(n) = F(n+1) - F(n) > 0 \quad (24)$$

or, combining the two conditions, when

$$\Delta F(n-1) < 0 < \Delta F(n) \quad (25)$$

From equation (22),

$$\Delta F(n) = \frac{P_{n+1}}{(1-V^{n+1})} - \frac{P_n}{(1-V^n)} \quad (26)$$

Upon expansion, equation (2.26) yields

$$\Delta F(n) = \frac{P(n+1) - P_{(n)} + V^{n+1}P_{(n)} - V^n P_{(n+1)}}{(1-V^{n+1})(1-V^n)} \quad (27)$$

But by definition,

$$P(n) = C + R_1 + VR_2 + V^2R_3 + \dots + V^{n-1}R_n \quad (28)$$

and

$$P(n+1) = C + R_1 + VR_2 + V^2R_3 + \dots + V^{n-1}R_n + V^n R_{n+1} \quad (29)$$

Thus

$$P_{(n+1)} - P_{(n)} = V^n R_{n+1} \quad (30)$$

Hence

$$V^{n+1}P_{(n)} = V^{n+1}C + V^{n+1}R_1 + V^{n+2}R_2 + V^{n+3}R_3 + \dots + V^{2n}R_n \quad (31)$$

$$V^n P_{(n+1)} = V^n C + V^n R_1 + V^{n+1}R_2 + V^{n+2}R_3 + \dots + V^{2n-1}R_n + V^{2n}R_{n+1} \quad (32)$$

Further

$$[V^{n+1}P_{(n)} - V^n P_{(n+1)}] = C(V^{n+1} - V^n) + (V^{n+1} - V^n) \sum_{i=1}^n V^{i-1}R_i - V^{2n}R_{n+1} \quad (33)$$

which, by re-arrangement,

$$= [(V^{n+1} - V^n)(C + \sum V^{i-1}R_i)] - V^{2n}R_{n+1} \quad (34)$$

but $(C + \sum V^{i-1}R_i) = P_{(n)}$, therefore

$$[V^{n+1}P_{(n)} - V^n P_{(n+1)}] = (V^{n+1} - V^n)P_{(n)} - V^{2n}R_{n+1} \quad (35)$$

By substituting equation (35) and (33) into equation (27) and re-arranging, we have:

$$\Delta F_{(n)} = \frac{V^n R_{n+1}(1-V^n) + (V^{n+1} - V^n)P_n}{(1-V^{n+1})(1-V^n)} \quad (36)$$

which, by further simplification and re-arrangement, gives

$$\Delta F_{(n)} = \frac{V^n}{1-V^{n+1}} \left(R_{n+1} - \frac{V-1}{1-V^n} P_n \right) \quad (37)$$

By definition, $V < 1$, hence equation (37) can be rewritten as

$$\Delta F_{(n)} = \frac{V^n}{1-V^{n+1}} \left(R_{n+1} - \frac{1-V}{1-V^n} P_n \right) \quad (38)$$

Considering equation (26), we expression (38) in this manner

$$\Delta F_{(n)} = \frac{V^n}{1-V^{n+1}} \left(R_{n+1} - \frac{1-V}{1-V^n} P_n \right) > 0 \quad (39)$$

Since $V < 1$, $\frac{V^n}{1-V^{n+1}}$ in equation (39) is always positive.

Hence the inequality in (39) still holds without it. So we drop it and we are left with the result:

$$\Delta F_{(n)} = \frac{V^n}{1-V^{n+1}} P_n > 0 \quad (40)$$

Or, by multiplying each term by $\frac{1-V^n}{1-V}$, we have:

$$\frac{1-V^n}{1-V} R_{n+1} - P_n > 0 \quad (41)$$

Or

$$\frac{1-V^n}{1-V} R_{n+1} > P_n \quad (42)$$

Or

$$R_{n+1} > P_n \frac{1-V}{1-V^n} \quad (43)$$

Equation (42) or (43) represents the first condition in equation (26) that $\Delta F_n > 0$. The second condition that $\Delta F_n < 0$ can similarly be derived and shown to be

$$\frac{1-V^{n-1}}{1-V} R_n - P_{n-1} < 0 \quad (44)$$

or,

$$\frac{1-V^{n-1}}{1-V} R_n < P_{n-1} \quad (45)$$

Equations (42) and (45) are the two basic formulae that are jointly required in the identification of optimum replacement unbound horizon at period n . They are interpreted in the following ways:

(1) As long as relation (45) holds, it is not time to replace the equipment. However, the equipment should be replaced in period n in which the relation (26) holds. That is, we replace only when

$$\frac{1-V^n}{1-V} R_{n+1} > P_n$$

And, further, since equation (45) will continue to hold until equation (48) holds, it is only equation (42) which we really have to look out for in identifying an optimal replacement period. Finally, we noted that equations (42) and (43) are equivalent and, as such, either of them can be applied.

Illustration

The purchase value of a machine is #10,000.00. The running/ maintenance cost is estimated at #1,000.00 per annum for the first five-years, increasing by #300 in the sixth and the subsequent years. If the prevailing rate of interest is 10 percent per annum:

(1) When is the optimum time (n) to replace the machine?

(2) What amount of money should be set aside each year for the replacement of the machine with a new one?

Computational Solution

With the application of equation (42), the model gives Table 1 as solution. It shows that the machine should be replaced at the end of 11 years:

Table 1: Solution to the Replacement machine

Year (<i>t</i>)	<i>C</i> Initial cost	<i>R_t</i> Maint enanc e cost	$1-V$ Discoun t cost	$1-V^n$ nth cost factor	V^{t-1} (<i>t</i> -1) period cost factor	$C+V^{t-1}R_t$ present maintenan ce cost	$P_n = C\sum V^{t-1}R_t$ present Total (cumulative) cost value	$\frac{(1-V^{n+1})}{(1-V)}R_{t+1}$ Present cumulative maintenance cost value
1	10000	1000	0.0909	0.0909	1.0000	11000.00	11,000.00	1000.00
2	0	1000	0.0909	0.1736	0.9091	909.10	11,909.10	1909.79
3	0	1000	0.0909	0.2487	0.8264	826.40	12,375.50	2735.97
4	0	1000	0.0909	0.3170	0.7513	751.30	13,486.80	3487.35
5	0	1000	0.0909	0.3791	0.6830	683.00	14,169.80	5421.67
6	0	1300	0.0909	0.4355	0.6209	807.17	14,976.97	7665.57
7	0	1600	0.0909	0.4868	0.5645	903.20	15,880.17	10175.14
8	0	1900	0.0909	0.5335	0.5132	975.08	16,855.25	12911.99
9	0	2200	0.0909	0.5759	0.4665	1026.30	17,881.55	15838.83
10	0	2500	0.0909	0.6145	0.4241	1060.25	18,941.80	18928.49
11*	0	2800	0.0909	0.6495	0.3855	1079.40	20,021.20	22150.17
12	0	3100	0.0909	0.6814	0.3505	1086.55	21,107.75	25486.91
13	0	3400	0.0909	0.7103	0.3186	1083.24	22190.99	

An alternative replacement model for the machine may be computed using equation (43):

$$R_{n+1} > P_n \frac{1-V}{1-V^n}$$

Therefore, an alternative Computation is given in Table 2 as follows:

Table 2: Alternative Iterative Computation

<i>Year</i> (<i>t</i>)	V^n nth Discou nt factor	$1 - V^n$ nth cost facto r	$1 - V$ Discou nt Cost (consta nt)	$\frac{(1 - V)}{(1 - V^n)}$ Amortizati on factor	$P_n \frac{(1 - V)}{(1 - V^n)}$ Present cumulati ve cost	R_{n+1} Present Cumulati ve maintena nce cost
1	0.9091	0.909	0.909	1.000	11000.00	1000
2	0.8264	0.173 6	0.0909	0.5236	6235.60	1000
3	0.7513	0.248 7	0.0909	0.3655	4654.83	1000
4	0.6830	0.317 0	0.0909	0.2868	3868.01	1000
5	0.6209	0.379 1	0.0909	0.2398	3397.92	1000
6	0.5645	0.435 5	0.0909	0.2087	3125.69	1300
7	0.5132	0.486 8	0.0909	0.1867	2964.83	1600
8	0.4665	0.533 5	0.0909	0.1703	2870.45	1900
9	0.4241	0.575 9	0.0909	0.1578	2821.71	2200
10	0.3855	0.614 5	0.0909	0.1479	2801.49	2500
11	0.3505	0.681 4	0.0909	0.1340	2682.84	2800
12	0.3186	0.710 3	0.0909	0.1334	2815.77	3100
13	0.2897	0.736 7	0.0909	0.1280	2840.45	3400

In order to develop Inventory model over an unbounded horizon, we let:

x , be defined as the production quantity

$C(x)$, be cost of producing quantity, x , in each period,

$h(j)$, the cost of holding the inventory at the end of period

d , a constant demand at each period

α , be the one-period discount factor; $0 \leq \alpha < 1$.

Then the appropriate dynamic programming recursion; or the present value for a finite horizon can be defined as:

$$f_n(i) = \min_x [C(x) + h(x + i - d) + \alpha f_{n-1}(x + i - d)]; 0 \leq \alpha < 1 \quad (46)$$

Equation (46) is expressible as an extremal function as

$$f(i) = \min_x [C(x) + h(x + i - d) + \alpha f(x + i - d)]; 0 \leq \alpha < 1 \quad (47)$$

Equation (47) expresses a system of equations over an unbounded horizon. In equation (47), $f(i)$ is conceived as the present value of an optimal policy over an unbounded horizon, given the inventory entering the current period is i . So equation (47) is now our new inventory model.

Hitherto, the inventory equation in use is:

$$\text{Minimize } \sum_{t=1}^N C_t(x_t, i_t) \quad 48$$

Where:

x_t = production quantity in period t

i_t = inventory at the end of period t

$C_t(x_t, i_t)$ = cost of incurring quantity x at period t which depends only on the production quantity x_t and the ending inventory level i_t , and on period t.

Now, the inventory model (48) only balanced off manufacturing with inventory holding costs. The economic cost of varying production levels from one period to the next was ignored and the model equally ignored discount factor knowing fully that money depreciates with time. All these deficiencies are taken care by our newly developed inventory model.

Discussion of Results

From Table 1, the optimal period $n=11$ and is starred * since it is at this period of time, that the present cumulative maintenance cost is higher than the present cumulative cost which is the optimal value amount that shall be needed on the 11th year. The actual money to set aside during the twelve years:

$$\begin{aligned}
\alpha_{11} &= \frac{(1-V)}{(1-V^{11})} P_{11} \\
&= \left[\frac{0.0909}{0.6495} \right] * 20021.20 \\
&= 0.13995381 * 20021.20 \\
&= \$2802.04 \text{ per annum}
\end{aligned}$$

However, from Table 2, the optimal period is $n = 11$ since at this period of time, it is observed that the present cumulative maintenance cost is higher than the present cumulative cost. At this point in time, it is advised that the machine or whatever is being maintained; be stopped. Any further maintenance would result to loss, so the maintaining article should be replaced with a new one.

Conclusion

This paper has been quite illuminating. Many computational models based on finite and infinite models; known respectively as bounded and unbounded horizons, have been numerically computed through various models. We developed iterative method for replacement and maintenance process using inventory model and they are very efficient even for large systems unlike the existing exhaustive enumeration algorithm which can be used only if the number of stationary policies is reasonably small.

Recommendations

Due to the importance of inventory in business, we developed and applied deterministic inventory model. So, we recommend that interested researchers should consider the case of developing probabilistic inventory model over unbounded horizon.

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