International Journal of Computing and Optimization
Vol. 3, 2016, no. 1, 71 - 82
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijco.2016.635

The Extended Balanced Truncation Algorithm

Cong Huu Nguyen

Thai Nguyen University
Tan Thinh, Thai Nguyen City, Viet Nam

Abstract

Model order reduction is a research direction which has attracted more interest to scientists in recent years. Actually there are many researches on model reduction for higher-order linear system but those on model reduction models for unstable higher-order linear system are still limited and existed many disadvantages.
This paper presents extended balanced truncation algorithm for unstable linear system. At the same time, to complete extended balanced truncation algorithm for unstable linear system, the author gives one definition and two new theorems with adequate proofs to determine the upper bound formula of order reduction error thereby the algorithm can automatically reduce order of linear unstable system based on that formula. The illustration shows the correctness of the model order algorithm.

Keywords: Model order reduction, extended balanced truncation algorithm, order reduction error

I. Introduction

There are two basic approaches to reduce order for unstable linear system as follows:

The first approach in [1] (indirectly order reduction (OR) method for unstable system) analyses the unstable system into sum of stable part and unstable part, and then, the OR algorithm such as balanced truncation [4] is applied to reduce order stable part. The new OR system is formed as the sum of order reduction part of stable system and unstable part. Under this approach, the effectiveness of order reduction depends mainly on order reduction algorithm applied to the stable system. However, in this approach the unstable system cannot be deleted in the OR system so the OR system always has higher order than the unstable one, which means this approach may not provide an enough good OR system to in-out relation of the original system in which unstable part occupies majority. But in reality, unstable parts often occupy a small share in the original system so this approach can give good OR results.

In the second approach (directly OR method for unstable system), the order of the unstable system is directly reduced under the extended balanced truncation algorithm of Zhou [7], algorithm of LQG [6], elemental analysis [2], applied balanced truncation algorithm of Boess [5], and extended balanced truncation of Zilochian [3]. The advantage of this approach is that order of OR system does not depend on order of the unstable system, which means order of this OR system can be lower than the unstable one's. However, algorithms following these methods have weaknesses as follows:

The extended balanced truncation algorithm of Zhou [7] need to address 4 Lyapunov equations so the computational complexity is high and it cannot be applied to all unstable systems and cannot give upper bound formula of order error reduction (OER).

The elemental analysis algorithm [2] requires experience of users and cannot give upper bound formula of OER.

The applied balanced truncation algorithm of Boess [5] has upper bound formula of OER but it only can be applied for discrete systems.

Study no. [3] has proposed mapping (transaction axes method) based on the value of the real part β of unstable poles having the largest real part value to switch unstable system into stable one, thereafter applying balanced truncation algorithm to reduce order. However, this algorithm has not given upper bound formula of OER so it cannot perform automatic reducing based on upper bound formula of OER.

According to these analyses, the article will focus on completing the study on the OR algorithm for unstable systems under the second approach (directly OR unstable system method), namely the research on determining upper bound formula of OER of extended balanced truncation algorithm of Zilochian [3] so that the algorithm can perform automatic reduce order based on upper bound formula of OER.

II. Extended balanced truncation algorithm

2.1 Problem of order reduction model

A multiple-input and output linear system is given with continuous-time constant parameters described in space stated in the following equations:

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}u \\ y &= \mathbf{C}x \end{aligned} \tag{2.1}$$

In which, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, $\mathbf{A} \in \mathbb{R}^{nxn}$, $\mathbf{B} \in \mathbb{R}^{nxp}$, $\mathbf{C} \in \mathbb{R}^{qxn}$

The goal of the order reduction problem with model described by (2.1) is to find models described by systems of equations:

$$\begin{aligned} \dot{x}_r &= \mathbf{A}_r x_r + \mathbf{B}_r u \\ y_r &= \mathbf{C}_r x_r \end{aligned} \tag{2.2}$$

In which, $x_r \in \mathbb{R}^r$, $u_r \in \mathbb{R}^p$, $y_r \in \mathbb{R}^q$, $\mathbf{A}_r \in \mathbb{R}^{rxr}$, $\mathbf{B}_r \in \mathbb{R}^{rxp}$, $\mathbf{C}_r \in \mathbb{R}^{qxr}$ and $r \ll n$, so that the model described by (2.2) can be replaced by the model described in (2.1) to be applied in analysis, design and control system.

2.2. Extended balanced truncation algorithm

The idea of the extended balanced truncation algorithm of Zilochian [3] is carried out mapping (the origin shift) to convert the original unstable system to stable form, then reduce order for stable system according to the extended balanced truncation algorithm obtaining the stable system. Finally, the algorithm performs backward projection (reverse the origin shelf) to convert the stable system to the unstable form similar to the original system.

The detailed content of the extended balanced truncation algorithm of Zilochian [3] is described as follows:

Algorithm 2.2. The extended balanced truncation algorithm of Zilochian [3]

Input: The system $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ is described as (2.1) (Unstable system). The transfer function of system (2.1) is given by $\mathbf{G}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.

Step 1: Identify the largest unstable pole α of system (2.1). Set $\beta = \text{real}(\alpha) + \delta,$ where $\delta \in \mathbb{R}$ arbitrarily small and $\delta > 0$.

Step 2: Convert the system $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ according to the following equations

$$\mathbf{A}_\beta = \mathbf{A} - \beta\mathbf{I},$$
$$\mathbf{B}_\beta = \mathbf{B},$$
$$\mathbf{C}_\beta = \mathbf{C}.$$

Step 3: Compute observability Gramian $\mathbf{Q}_\beta$ and controllability Gramian $\mathbf{P}_\beta$ of system $(\mathbf{A}_\beta,\ \mathbf{B}_\beta,\ \mathbf{C}_\beta)$ from two Lyapunov equations:

$$\mathbf{A}_\beta \mathbf{P}_\beta + \mathbf{P}_\beta \mathbf{A}_\beta^{\mathrm{T}} = -\mathbf{B}_\beta \mathbf{B}_\beta^{\mathrm{T}},$$
$$\mathbf{A}_\beta^{\mathrm{T}} \mathbf{Q}_\beta + \mathbf{Q}_\beta \mathbf{A}_\beta = -\mathbf{C}_\beta^{\mathrm{T}} \mathbf{C}_\beta.$$

Step 4: Compute Cholesky factorization of matrix $\mathbf{P}_\beta = \mathbf{R}_{\beta p} \mathbf{R}_{\beta p}^{\mathrm{T}}$, where $\mathbf{R}_{\beta p}$ is upper triangle matrix.

Step 5: Compute Cholesky factorization of matrix $\mathbf{Q}_\beta = \mathbf{R}_{\beta o} \mathbf{R}_{\beta o}^{\mathrm{T}}$, where $\mathbf{R}_{\beta o}$ is upper triangle matrix.

Step 6: Compute SVD of matrix $\mathbf{R}_{\beta o} \mathbf{R}_{\beta p}^{\mathrm{T}} = \mathbf{U}_\beta \mathbf{\Lambda} \mathbf{V}_\beta^{\mathrm{T}}$.

Step 7: Compute nonsingular transformation $\mathbf{T}_\beta$

$$\mathbf{T}_\beta^{-1} = \mathbf{R}_{\beta p} \mathbf{V}_\beta \mathbf{\Lambda}_\beta^{-1/2}.$$

Step 8: Compute

$$\left(\hat{\mathbf{A}}_{\beta},\ \hat{\mathbf{B}}_{\beta},\ \hat{\mathbf{C}}_{\beta}\right)=\left(\mathbf{T}_{\beta}^{-1}\mathbf{A}_{\beta}\mathbf{T}_{\beta},\ \mathbf{T}_{\beta}^{-1}\mathbf{B}_{\beta},\ \mathbf{C}_{\beta}\mathbf{T}_{\beta}\right).$$

Step 9: Choose re-order r so that r << n.

Step 10: $\left(\hat{\mathbf{A}}_{\beta},\ \hat{\mathbf{B}}_{\beta},\ \hat{\mathbf{C}}_{\beta}\right)$ is partitioned as follows::

$$\hat{\mathbf{A}}_{\beta}=\begin{bmatrix}\hat{\mathbf{A}}_{11\beta} & \hat{\mathbf{A}}_{12\beta}\\ \hat{\mathbf{A}}_{21\beta} & \hat{\mathbf{A}}_{22\beta}\end{bmatrix},\ \hat{\mathbf{B}}_{\beta}=\begin{bmatrix}\hat{\mathbf{B}}_{1\beta}\\ \hat{\mathbf{B}}_{2\beta}\end{bmatrix},\ \hat{\mathbf{C}}_{\beta}=\begin{bmatrix}\hat{\mathbf{C}}_{1\beta} & \hat{\mathbf{C}}_{2\beta}\end{bmatrix},$$

where $\hat{\mathbf{A}}_{11\beta}\in\mathbb{R}^{rxr}$, $\hat{\mathbf{B}}_{1\beta}\in\mathbb{R}^{rxp}$, $\hat{\mathbf{C}}_{1\beta}\in\mathbb{R}^{qxr}$.

We obtain reduced stable system $\left(\hat{\mathbf{A}}_{11\beta},\ \hat{\mathbf{B}}_{1\beta},\ \hat{\mathbf{C}}_{1\beta}\right)$.

Step 11: Convert reduced stable system $\left(\hat{\mathbf{A}}_{11\beta},\ \hat{\mathbf{B}}_{1\beta},\ \hat{\mathbf{C}}_{1\beta}\right)$ according to the following equations:

$$\hat{\mathbf{A}}_{11}=\hat{\mathbf{A}}_{11\beta}-\beta\mathbf{I},$$
$$\hat{\mathbf{B}}_{1}=\hat{\mathbf{B}}_{1\beta},$$
$$\hat{\mathbf{C}}_{1}=\hat{\mathbf{C}}_{1\beta}.$$

Output: The reduced system $\left(\hat{\mathbf{A}}_{11},\ \hat{\mathbf{B}}_{1},\ \hat{\mathbf{C}}_{1}\right)$

2.3. Completing extended balanced truncation algorithm

The extended balanced truncation algorithm has not given upper bound formula of OER so it cannot perform automatic reduction based on upper bound formula of OER. To complete this algorithm, the authors will identify and prove the correctness of the upper bound formula of OER of this algorithm.

The transfer function of system (2.1) is given by $\mathbf{G}(s)=\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}$

Definition 1. The system (2.1) is called -β stable if $\text{real}(\lambda(\mathbf{A}))<\beta$, with β is a non-negative real numbers ($\beta\geq 0$). The set of continuous β - stable system is denoted by C_{β}. The $H_{\infty,\beta}$-norm of $\mathbf{G}(s)\in C_{\beta}$ is denoted by:

$$\left\|\mathbf{G}(s)\right\|_{H_{\infty,\beta}}:=\sup_{real(\lambda(\mathbf{A}))<\beta}\sigma_{\max}(\mathbf{G}(s))$$
$$=\sup_{\omega\in\mathbb{R}}\sigma_{\max}(\mathbf{G}(\beta+j\omega)).$$

Where $\sigma_{\max}(\mathbf{G}(s))$ is the largest singular value of $\mathbf{G}(s)$.

In the case that $\beta=0$ the system (1) is called asymptotically stable as ordinary meaning [1]. Matrix $\mathbf{A}$ in this case is Hurwitz matrix, i.e., $\left|\text{real}(\lambda(\mathbf{A}))\right|<0$, and the $H_{\infty,\beta}$-norm of $\mathbf{G}(s)$ coincide with the H_{∞}-norm of $\mathbf{G}(s)$ as usual meaning

$$\left\|\mathbf{G}(s)\right\|_{H_{\infty,0}}=\left\|\mathbf{G}(s)\right\|_{H_{\infty}}=\sup_{\omega\in\mathbb{R}}\sigma_{\max}\left(\mathbf{G}\left(j\omega\right)\right).$$

Set $\hat{\mathbf{G}}_{1\beta}(s) = \hat{\mathbf{C}}_{1\beta}\left(s\mathbf{I} - \hat{\mathbf{A}}_{11\beta}\right)^{-1}\hat{\mathbf{B}}_{1\beta}$ is the matrix transfer function of system $\left(\hat{\mathbf{A}}_{11\beta},\ \hat{\mathbf{B}}_{1\beta},\ \hat{\mathbf{C}}_{1\beta}\right)$ and $\hat{\mathbf{G}}_{1}(s) = \hat{\mathbf{C}}_{1}\left(s\mathbf{I} - \hat{\mathbf{A}}_{11}\right)^{-1}\hat{\mathbf{B}}_{1}$ is the matrix transfer function of reduced system $\left(\hat{\mathbf{A}}_{11},\ \hat{\mathbf{B}}_{1},\ \hat{\mathbf{C}}_{1}\right)$ in **Algorithm 2.2**, we have two theorems as follows:

Theorem 1: For any continuous system $\mathbf{G}(s) \in \mathbb{C}_{\beta}$ represented by (2.1), we consider the system $\mathbf{G}_{\beta}(s)$ with realization $\left(\mathbf{A}_{\beta},\ \mathbf{B}_{\beta},\ \mathbf{C}_{\beta}\right) = \left(\mathbf{A} - \beta\mathbf{I},\ \mathbf{B},\ \mathbf{C}\right)$. Then, the following things hold:

(i) $\mathbf{G}_{\beta}$ is asymptotically stable,

(ii) The H_{∞}-norm of $\mathbf{G}_{\beta}(s)$ is equal to the $\mathrm{H}_{\infty,\beta}$-norm of $\mathbf{G}(s)$, …

$$\left\|\mathbf{G}_{\beta}(s)\right\|_{\mathrm{H}_{\infty}} = \left\|\mathbf{G}(s)\right\|_{\mathrm{H}_{\infty,\beta}}.$$

Proof of Theorem 1:

(i) From $\mathbf{G}(s) \in \mathbb{C}_{\beta}$, we have $\mathrm{real}(\lambda(\mathbf{A})) < \beta,\ \beta \geq 0$ and $\mathrm{real}(\lambda(\mathbf{A} - \beta\mathbf{I})) < 0$.

(ii) It holds that

$$\begin{aligned}\mathbf{G}_{\beta}(j\omega) &= \mathbf{C}_{\beta}(j\omega\mathbf{I} - \mathbf{A}_{\beta})^{-1}\mathbf{B}_{\beta}\\ &= \mathbf{C}(j\omega\mathbf{I} - \mathbf{A} + \beta\mathbf{I})^{-1}\mathbf{B}\\ &= \mathbf{C}((\beta + j\omega)\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\\ &= \mathbf{G}(\beta + j\omega),\end{aligned}$$

and thus

$$\begin{aligned}\left\|\mathbf{G}_{\beta}\right\|_{\mathrm{H}_{\infty}} &= \sup_{\omega\in\mathbb{R}} \sigma_{\max}(\mathbf{G}_{\beta}(j\omega))\\ &= \sup_{\omega\in\mathbb{R}} \sigma_{\max}(\mathbf{G}(\beta + j\omega))\\ &= \left\|\mathbf{G}\right\|_{\mathrm{H}_{\infty,\beta}}.\end{aligned}$$

□

Theorem 2: Let $\mathbf{G}(s) \in \mathbb{C}_{\beta}$ and $\hat{\mathbf{G}}_{1}(s)$ be reduced order systems obtained as in Algorithm 2.2. Then the following bound for the error system hold:

$$\left\|\mathbf{G}(s) - \hat{\mathbf{G}}_{1}(s)\right\|_{H_{\infty,\beta}} \leq 2\left(\sigma_{r+1} + \ldots + \sigma_{n}\right),$$

where $\sigma_{1},\ldots,\sigma_{n}$ are the Hankel singular values of $\mathbf{G}_{\beta}(s)$.

Proof of Theorem 2: Let

$\mathbf{E}(s) = \mathbf{G}(s) - \hat{\mathbf{G}}_{1}(s) = \mathbf{C}_{e}\left(s\mathbf{I} - \mathbf{A}_{e}\right)^{-1}\mathbf{B}_{e}$, we have

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \hat{\mathbf{A}}_{11} \end{bmatrix}, \mathbf{B}_e = \begin{bmatrix} \mathbf{B} \\ \hat{\mathbf{B}}_1 \end{bmatrix}, \mathbf{C}_e = \begin{bmatrix} \mathbf{C} & -\hat{\mathbf{C}}_1 \end{bmatrix}.$$

From $\text{real}(\lambda(\mathbf{A})) < \beta$, $\text{real}(\lambda(\hat{\mathbf{A}}_{11})) < \beta$, we have $\mathbf{E}(s) \in \mathbb{C}_\beta$. Using Theorem 1 we obtain that

$$\left\|\mathbf{E}(s)\right\|_{\mathrm{H}_{\infty,\beta}} = \left\|\mathbf{E}_\beta(s)\right\|_{\mathrm{H}_\infty} = \left\|\mathbf{G}_\beta(s) - \hat{\mathbf{G}}_{1\beta}(s)\right\|_{\mathrm{H}_\infty},$$

where $\mathbf{E}_\beta(s) = \mathbf{C}_e\left(s\mathbf{I} - \left(\mathbf{A}_e - \beta\mathbf{I}\right)\right)^{-1}\mathbf{B}_e$. Since system $\mathbf{G}_\beta(s)$ and $\hat{\mathbf{G}}_{1\beta}(s)$ are asymptotically stable and $\hat{\mathbf{G}}_{1\beta}(s)$ is reduced system obtained by balanced truncation of $\mathbf{G}_\beta(s)$, we get that

$$\left\|\mathbf{G}_\beta(s) - \hat{\mathbf{G}}_{1\beta}(s)\right\|_{\mathrm{H}_\infty} \le 2\left(\sigma_{r+1} + \ldots + \sigma_n\right),$$

where $\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of $\mathbf{G}_\beta(s)$. □

III. The illustrative example

3.1. Reducing unstable system

We have a high order unstable system as follows:

$$\mathbf{S}(s) = \frac{\mathbf{H}(s)}{\mathbf{D}(s)} \tag{3.1}$$

$$\mathbf{H}(s) = -1.26s^{15} - 110.4s^{14} - 3959s^{13} - 8.089.10^4 s^{12} - 1.078.10^6 s^{11} - 1.006.10^7 s^{10}$$
$$-6.869.10^7 s^9 - 3.547.10^8 s^8 - 1.419.10^9 s^7 - 4.478.10^9 s^6 - 1.116.10^{10} s^5 - 2.142.10^{10} s^4$$
$$-2.96.10^{10} s^3 - 2.616.10^{10} s^2 - 1.183.10^{10} s - 1.536.10^9$$

$$\mathbf{D}(s) = 0.000118s^{15} + 0.0202s^{14} + 1.051s^{13} + 22.89s^{12} + 222.5s^{11} + 165.6s^{10} - 1.99.10^4 s^9$$
$$-2.433.10^5 s^8 - 1.533.10^6 s^7 - 5.942.10^6 s^6 - 1.438.10^7 s^5 - 2.042.10^7 s^4 - 1.401.10^7 s^3$$
$$-2.108.10^6 s^2 - 1.49.10^{-8} s$$

Performing reduced order model unstable system (3.1) by extended balanced truncation algorithm, the results are shown in the following:

Step 1: Identify the largest unstable pole α of system (3.1). We obtained $\alpha = 10,313$. Set $\beta = \text{real}(\alpha) + \delta = 10,323$

Step 2**:** Convert the unstable system $\mathbf{S}(s)$ to stable system $\mathbf{S}_\beta(s)$ according to the following equations

$$A_\beta = A - \beta I = \begin{bmatrix} -181.5308 & -69.2774 & -23.5797 & -14.3273 & -1.3331 & 10.0562 & 7.6510 & 6.0266 & 2.9194 & 1.7667 & 1.2539 & 0.8603 & 0.5178 & 0.0000 & 0 \\ 128 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 160 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -10.3233 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -10.3233 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -10.3233 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10.3233 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & -10.3233 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & -10.3233 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & -10.3233 \end{bmatrix},$$

$$B_\beta = B = \begin{bmatrix} 8192 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_\beta = C = 1.0e^4 \left[0.0108\ \ 0.0058\ \ 0.0020\ \ 0.0010\ \ -0.0008\ \ -0.0017\ \ -0.0011\ \ -0.0009\ \ -0.0004\ \ -0.0002\ \ -0.0002\ \ -0.0001\ \ -0.0001\ \ -0.0182\ \ -1.2074 \right],$$

$$D_\beta = D = \left[-1.0632e^4 \right]$$

***Step 3-10*:** Performing reduced order model system follows step 3 – 10, the results are shown in the following table:

Table 1. Results of the order reduction of stable system $\mathbf{S}_\beta(s)$

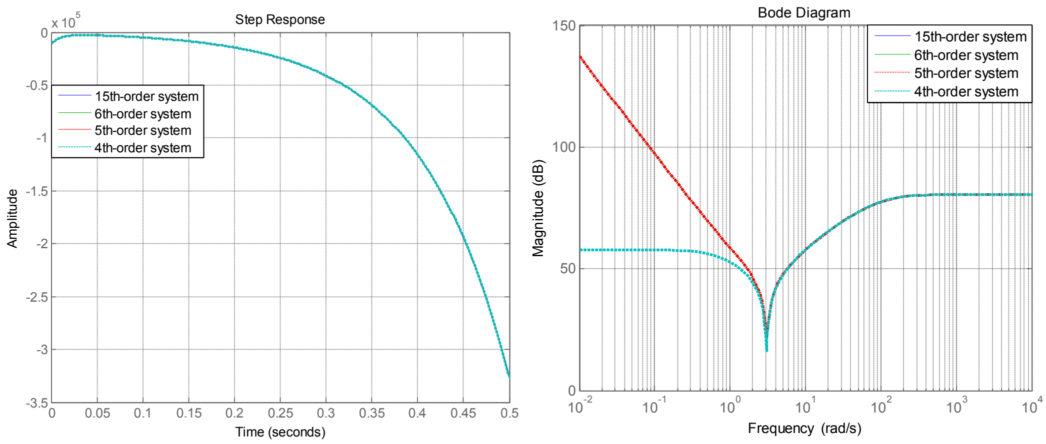
Order	Reduced order system $\hat{\mathbf{S}}_{1\beta}(s)$	Error $\left\Vert \mathbf{S}_\beta(s) - \hat{\mathbf{S}}_{1\beta}(s) \right\Vert_{H_\infty}$
6	$\dfrac{-1.063.10^4 s^6 - 1.092.10^6 s^5 - 4.207.10^7 s^4 - 8.124.10^8 s^3 - 8.488.10^9 s^2 - 4.612.10^{10} s - 1.027.10^{11}}{s^6 + 186.3 s^5 + 1.008.10^4 s^4 + 2.026.10^5 s^3 + 1.724.10^6 s^2 + 5.301.10^6 s + 5.283.10^4}$	$3.484.10^{-7}$
5	$\dfrac{-1.063.10^4 s^5 - 9.871.10^5 s^4 - 3.231.10^7 s^3 - 4.928.10^8 s^2 - 3.612.10^9 s - 1.038.10^{10}}{s^5 + 176.4 s^4 + 8335 s^3 + 1.201.10^5 s^2 + 5.352.10^5 s + 5340}$	$3.5.10^{-7}$
4	$\dfrac{-1.063.10^4 s^4 - 8.665.10^5 s^3 - 2.256.10^7 s^2 - 2.397.10^8 s - 9.246.10^8}{s^4 + 165.1 s^3 + 6470 s^2 + 4.765.10^4 s + 475.8}$	0.0346
3	$\dfrac{-1.063.10^4 s^3 - 3.28.10^5 s^2 - 3.174.10^6 s - 7.74.10^6}{s^3 + 114.5 s^2 + 399.5 s + 3.983}$	10.15

Step 11 : Convert reduced stable system $\mathbf{S}_{1\beta}(s)$ to the reduced β-stable system $\hat{\mathbf{S}}_1(s)$, the results are presented in table 2:

Table 2. Results of the order reduction of $\mathbf{S}(s)$

Order	Reduced order system $\hat{\mathbf{S}}_1(s)$	Error $\left\| \mathbf{S}(s) - \hat{\mathbf{S}}_1(s) \right\|_{H_{\infty,\beta}}$
6	$\dfrac{-1.063.10^4 s^6 - 4.337.10^5 s^5 - 2.689.10^6 s^4 - 5.225.10^7 s^3 - 2.36.10^7 s^2 - 1.503.10^7 s + 1.135.10^7}{s^6 + 124.4s^5 + 2063s^4 - 3.711.10^4 s^3 + 1.557.10^4 s^2 + 1.65.10^{-5} s - 4.41.10^{-6}}$	$3.484.10^{-7}$
5	$\dfrac{-1.063.10^4 s^5 - 4.383.10^5 s^4 - 2.877.10^6 s^3 - 6.463.10^6 s^2 - 2.638.10^7 s - 2.638.10^7}{s^5 + 124.8s^4 + 2117s^3 - 3.62.10^4 s^2 - 1.45.10^{-5} s + 1.027.10^{-5}}$	$3.5.10^{-7}$
4	$\dfrac{-1.063.10^4 s^4 - 4.274.10^5 s^3 - 2.522.10^6 s^2 - 4.157.10^6 s - 2.191.10^7}{s^4 + 123.8s^3 + 1998s^2 - 3.757.10^4 s + 2.789.10^4}$	0.0346
3	$\dfrac{-1.063.10^4 s^3 + 1317s^2 + 1.981.10^5 s + 1.772.10^6}{s^3 + 83.49s^2 - 1644s + 6978}$	10.15

Step response and bode plots of the original system (15th-order system), reduced order systems results are shown in Figure 1.



(a) *(b)*

Fig. 1. Step response (a) and bode plots (b) of the original system, reduced order systems

Comment: Through the result of the order reduction error and Figure 1, it can be seen that:

- The order reduction error under the standard $H_{\infty,\beta}$ of 5th, 6th-order system is small; the step response and bode plot of 5th, 6th-order system completely coincides with the ones of the original system.

- The order reduction error inder the standard $H_{\infty,\beta}$ of 4th-order system is much more than the order reduction error of the 5th, 6th-order system; the step response of 4th-order system completely coincides with the ones of the original system; bode plots of 4th-order sytem is deviation from the bode plots of the original system in the low frequency (less than 1.98 rad/s).

Therefore, depending on the order reduction error request and the application range of the model order reduction in specific problem,we can choose the corresponding model order reduction for the original system. Especically, if we want the order reduction system having the lowest possible order and mainly care about the false step response compared to the small original system, we can choose the 4th-order system instead of the original system. If we want the lowest order system but small order reduction error, the false step response and the false bode plots versus the small original system, we can choose the 5th-order system instead of the original system.

The above result shows that the extended balanced truncation algorithm can reduce order for high-order unstable linear system.

3.2. Reducing higher- order controller

Design of controller of the bicycle balanced system under H∞ sustainable control algorithm shown in detail in [8], H∞ full order controller is designed as follows:

$$\mathbf{R}(s)=\frac{\mathbf{H}(s)}{\mathbf{D}(s)} \tag{3.2}$$

$$\begin{aligned}\mathbf{H}(s)=&-2.23.10^{-7}s^{30}-4.67.10^{-4}s^{29}-0.266s^{28}-22.96s^{27}-1006s^{26}-2.853.10^{4}s^{25}\\&-5.837.10^{5}s^{24}-9.144.10^{6}s^{23}-1.139.10^{8}s^{22}-1.158.10^{9}s^{21}-9.776.10^{9}s^{20}\\&-6.949.10^{10}s^{19}-4.199.10^{11}s^{18}-2.172.10^{12}s^{17}-9.663.10^{12}s^{16}-3.71.10^{13}s^{15}\\&-1.231.10^{14}s^{14}-3.53.10^{14}s^{13}-8.74.10^{14}s^{12}-1.862.10^{15}s^{11}-3.398.10^{15}s^{10}\\&-5.276.10^{15}s^{9}-6.903.10^{15}s^{8}-7.511.10^{15}s^{7}-6.676.10^{15}s^{6}-4.721.10^{15}s^{5}\\&-2.556.10^{15}s^{4}-9.953.10^{14}s^{3}-2.482.10^{14}s^{2}-2.977.10^{13}s-0.00439\end{aligned}$$

$$\begin{aligned}\mathbf{D}(s)=&\,4.971.10^{-14}s^{30}+2.032.10^{-10}s^{29}+2.663.10^{-7}s^{28}+1.221.10^{-4}s^{27}+9.72.10^{-3}s^{26}\\&+0.3918s^{25}+10.14s^{24}+187.1s^{23}+2612s^{22}+2.862.10^{4}s^{21}+2.523.10^{5}s^{20}\\&+1.82.10^{6}s^{19}+1.088.10^{7}s^{18}+5.428.10^{7}s^{17}+2.273.10^{8}s^{16}+8.005.10^{8}s^{15}\\&+2.372.10^{9}s^{14}+5.9.10^{9}s^{13}+1.225.10^{10}s^{12}+2.107.10^{10}s^{11}+2.962.10^{10}s^{10}\\&+3.341.10^{10}s^{9}+2.941.10^{10}s^{8}+1.931.10^{10}s^{7}+8.743.10^{9}s^{6}+2.286.10^{9}s^{5}\\&+1.519.10^{8}s^{4}-5.226.10^{7}s^{3}+3.6.10^{-6}s^{2}+5.32.10^{-22}s\end{aligned}$$

Converting controller (3.2) to minimal realization and pole-zero cancellation, we obtain the following result:

$$\mathbf{R}_m(s)=\frac{\mathbf{H}_m(s)}{\mathbf{D}_m(s)} \tag{3.3}$$

$$
\begin{aligned}
\mathbf{H}_m(s) = &-4.485.10^6 s^{27} - 4.231.10^8 s^{26} - 1.912.10^{10} s^{25} - 5.51.10^{11} s^{24} - 1.138.10^{13} s^{23} \\
&- 1.794.10^{14} s^{22} - 2.245.10^{15} s^{22} - 2.29.10^{16} s^{20} - 1.939.10^{17} s^{19} - 1.381.10^{18} s^{18} \\
&- 8.359.10^{18} s^{17} - 4.33.10^{19} s^{16} - 1.929.10^{20} s^{15} - 7.413.10^{20} s^{14} - 2.462.10^{21} s^{13} \\
&- 7.066.10^{21} s^{12} - 1.75.10^{22} s^{11} - 3.732.10^{22} s^{10} - 6.814.10^{22} s^{9} - 1.058.10^{23} s^{8} \\
&- 1.385.10^{23} s^{7} - 1.508.10^{23} s^{6} - 1.341.10^{23} s^{5} - 9.487.10^{22} s^{4} - 5.138.10^{22} s^{3} \\
&- 2.001.10^{22} s^{2} - 4.99.10^{21} s - 5.987.10^{20}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}_m(s) = &\, s^{27} + 2088 s^{26} + 1.803.10^5 s^{25} + 7.485.10^6 s^{24} + 1.966.10^8 s^{23} + 3.66.10^9 s^{22} \\
&+ 5.14.10^{10} s^{21} + 5.657.10^{11} s^{20} + 5.003.10^{12} s^{19} + 3.618.10^{13} s^{18} + 2.166.10^{14} s^{17} \\
&+ 1.083.10^{15} s^{16} + 4.539.10^{15} s^{15} + 1.601.10^{16} s^{14} + 4.748.10^{16} s^{13} + 1.182.10^{17} s^{12} \\
&+ 2.456.10^{17} s^{11} + 4.226.10^{17} s^{10} + 5.944.10^{17} s^{9} + 6.709.10^{17} s^{8} + 5.908.10^{17} s^{7} \\
&+ 3.881.10^{17} s^{6} + 1.758.10^{17} s^{5} + 4.598.10^{16} s^{4} + 3.058.10^{15} s^{3} - 1.051.10^{15} s^{2}
\end{aligned}
$$

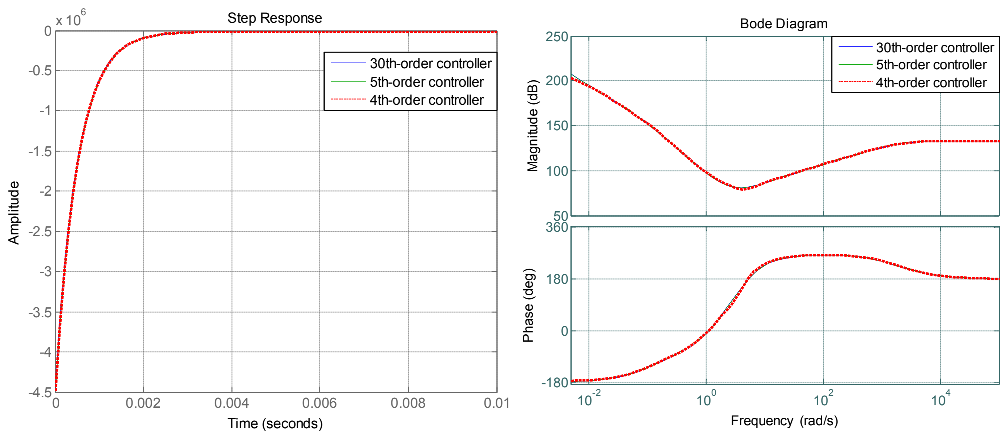
The controller $\mathbf{R}_m(s)$ is an unstable system because the controller $\mathbf{R}_m(s)$ has 3 positive poles which are 0; 0; 0.103.

Performing reduced order model unstable system (3.3) by extended balanced truncation algorithm, the results are shown in the following table:

Table 3. Results of the order reduction of controller $\mathbf{R}_m(s)$

Order	Reduced order controller $R_r(s)$
5	$\dfrac{-4.485.10^6 s^5 - 6.804.10^7 s^4 - 4.123.10^8 s^3 - 1.235.10^9 s^2 - 1.816.10^9 s - 1.09.10^9}{s^5 + 2009 s^4 + 1.833.10^4 s^3 - 1913 s^2 + 6.614.10^{-9} s - 8.44.10^{-10}}$
4	$\dfrac{-4.485.10^6 s^4 - 2.655.10^7 s^3 - 1.191.10^8 s^2 - 1.811.10^8 s - 1.182.10^8}{s^4 + 2000 s^3 - 205.6 s^2 - 0.1231 s + 0.003463}$

Step response and bode plots of the original controller (30th-order controller), reduced order controllers results are shown Figure 2.



(a) (b)

Fig. 2. Step response (a) and bode plots (b) of the original controller, reduced order systems

Remarks of the results: Step response and bode plots of the 4^{th}, 5^{th} - order controller almost coincides with step and bode plots of the original controller. Therefore, the 5^{th}, 4^{th} - order controller can be used to replace the original controller. The above results show that the higher-order controller (unstable system) can be reduced by extended balanced truncation algorithm.

IV. Conclusions

The article introduced the extended balanced truncation algorithm. Besides, the author has come up with one definication and two new theorems with the adequate proof to determine the upper bound fomula of the order reduction error in order to complete the extended balanced truncation algorithm for unstable system. The stimulation results showed the correctness of the upper bound fomula of the order reduction error and the introducted algorithm.

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Received: March 24, 2016; Published: May 4, 2016