International Journal of Computing and Optimization
Vol. 3, 2016, no. 1, 1 - 9
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijco.2016.5927

Duality Theorems for Non-Convex Mixed Integer Programming Problems

B. Sunita Mishra

Orissa Engineering College,Bhubaneswar, Odisha, India

J.R. Nayak

Department of Mathematics, Siksha O Anusandhan University
Bhubaneswar, Odisha, India

P. K. Satpathy

Department of Mathematics,Siksha O Anusandhan University
Bhubaneswar, Odisha, India

Abstract

In this paper, we consider maximin and minimax nonlinear mixed integer programming problems which are not symmetric in the duality sense. Under generalized conditions which involve increasing and decreasing functions, we compare the supremum infimum of the maximin problem with the infimum supremum of the minimax problem. The weak duality theorem for minimax and symmetric dual nonlinear mixed integer programming problem is derived as a particular case. The earlier results on minimax and symmetric duality in nonlinear mixed integer programming are thus generalized for monotonic functions. The results equally hold for strong pseudo convex and strong pseudo concave functions.

Keywords: Non linear Programming, Mixed integer Programming, Duality Theorems, Reproducing Cone, Monotonic functions, Strong pseudo convex functions

1 Introduction

Nonlinear dual problems are formulated basing on the conjugate functions[1], Lagrangian multipliers[2], minimax type [3], symmetric type [4]. We have presented the weak, the strong and the converse duality results for general non symmetric and minimax type problems using increasing and decreasing functions on a reproducing cone. Strong pseudo convex and strong pseudo concave functions equally satisfy the theorems on arbitrary cones.

Our results generalize the works of [5] and [6] who proved the same results under stronger assumptions on the cone and the functions. Convex and concave functions are considered in [5] on the non-negative orthant as the cone. The extension of these results to any arbitrary cone is presented in [6]. The results were further modified by [7] by assuming the functions to be pseudo-convex and pseudo concave. We have generalized the results with additional feasibility conditions in this presentation.

Classic results in nonlinear programming which cover duality and mixed integer programming can be seen in [8-12]. Recent developments in this area is vividly presented in[13-16]. Nonlinear mixed integer programming applied to different real life models are presented in [17-27]. The motivating results for our work are based on the most important analysis of symmetric duality in [28, 29].

1.1 Notations and terminologies

Let U and V be arbitrary sets of integers belonging to R^{n_1} and R^{m_1} respectively. Let C_1 and C_2 be reproducing cones with vertices at the origin with nonempty interiors in R^{n-n_1} and R^{m-m_1} respectively. The polar of C is defined as

$$C_i^* = \left\{x^t z \leq 0 \text{ for } x \in C_i \text{ where } x^t \text{ is the transpose of } x,\ i = 1, 2\right\}$$

Some of the components of x and y which belong to arbitrary sets of integers are being constrained. Let the first n_1 components of x and the first m_1 components of $y\left(0 \leq n_1 \leq n,\ 0 \leq m_1 \leq m\right)$ arbitrarily be integers.

Let $(x, y) = \left(x^1, x^2, y^1, y^2\right)$, $x^1 = \left(x_1, x_2, ...x_{n_1}\right)$ and $y^1 = \left(y_1, y_2, ..., y_{m_1}\right)$

Let K(x, y) be a twice differentiable real valued function defined on an open set in R^{n+m} containing $S \times T$ where $S = U \times C_1$ and $T = V \times C_2$. $\nabla_{x^2} K\left(\bar{x}, \bar{y}\right)$ denotes the gradient vector of K with respect to x^2 at the point $\left(\bar{x}, \bar{y}\right)$. $\nabla_{y^2} K\left(\bar{x}, \bar{y}\right)$ is defined similarly. $\nabla_{x^2 x^2} K\left(\bar{x}, \bar{y}\right)$ denotes the Hessian matrix of second partial derivatives with respect to x^2 evaluated at $\left(\bar{x}, \bar{y}\right)$. $\nabla_{x^2 y^2} K\left(\bar{x}, \bar{y}\right), \nabla_{y^2 x^2} K\left(\bar{x}, \bar{y}\right)$ and $\nabla_{y^2 y^2} K\left(\bar{x}, \bar{y}\right)$ and defined similarly. We say that K is increasing / decreasing on $C_1 \times C_2$ iff K is increasing in x^2 for each x^1, y and decreasing in y^2 for each x and y^1 i.e.

$K\left(x^2, y\right) \geq K\left(x^1, y\right)$ for each x^1, $\left(y \in C_2\right)$ and
$K\left(x, y^2\right) \leq K\left(x, y^1\right)$ for each $y^1\left(x \in C_1\right)$ respectively.

2. The Problems

Consider the following pair of nonlinear mixed integer programming problems:

(P_0) $\underset{x^2}{\text{Max}}\,\underset{x^2 y}{\text{Max}}\ f = K\left(x, y\right) - \lambda(y^2)t\nabla y^2 k\left(x, y\right)$

such that $x^2 \in U, \left(x^2, y\right) \in C_1 \times T, \nabla_{y^2} K\left(x, y\right) \in C_2^*, \lambda \geq 1$

(D_0) $\underset{y^2}{\text{Max}}\,\underset{x.y^2}{\text{Max}}\ g = K\left(x, y\right) - \mu\left(x^2\right)^t \nabla_{x^2} K\left(x, y\right)$

such that $y^1 \in V, \left(x, y^2\right) \in S \times C_2,\ \nabla_{x^2} K\left(x, y\right) \in C_1^*\ \mu \geq 1$

The set of feasible solutions of (P_0) and (D_0) are

$$P_1 = \left\{\left(x, y\right) \mid x^1 \in U, \left(x^2, y\right) \in C_1 XT, \nabla_{y^2} K\left(x, y\right) \in C_2^*, \lambda \geq 1\right\}$$

$$D_1 = \left\{\left(x, y\right) \mid y^2 \in V, \left(x^2, y\right) \in S\ X\ C_2, \nabla_{x^2} K\left(x, y\right) \in C_2^*, \mu \geq 1\right\}$$

respectively.

When the cone is the nonnegative orthant we see that for $\lambda = \mu = 1$ the problems (P_0), (D_0) reduce to the pair of problems (D), (P) of [5]. When the cone is arbitrary the pair of problems considered by [6] becomes a particular case of (P_0), (D_0).

3 Main Results

Our results have been derived under general assumptions of

(i) K(x,y) is increasing /decreasing on $C_1 \times C_2$

(ii) K(x,y) is separable with respect to x^1 or y^1, and

(iii) the existence of the feasible sets P_1 and D_1 having the properties: if $\left(x, y\right) \in P_1$ and $\left(u, v\right) \in D_1$ then $x^2 - u^2 \in C_1$ and $v^2 - y^2 \in C_2$ where C and C are reproducing cones.

Theorem. 1 (Weak Duality).

The sup inf of f(x, y) is greater than or equal to the inf sup of g(x, y) for any $\left(x, y\right) \in P_1$ and all $\left(x, y\right) \in D_1$.

Proof: Let $z_1 = \underset{x^1}{\max}\,\underset{x^2 y}{\min}\left\{f \mid \left(x, y\right) \in P_1\right\},\ w_1 = \underset{y^1}{\max}\,\underset{x.y^2}{\min}\left\{f \mid \left(x, y\right) \in P_1\right\}$ (3.1)

Since K(x, y) is separable with respect to x^1, we have

$$K\left(x, y\right) = K^1\left(x^1\right) + K^2\left(x^2, y\right) \tag{3.2}$$

The same holds if K (x, y) is separable with respect to y^1. Then z_1 can be written as

$$z_1 = \max_{x^1} \min_{x^2 . y} \left\{ K^1\left(x^1\right) + K^2\left(x^2, y\right) - \lambda\left(y^2\right)^t \nabla_{y^2}\left(K^1\left(x^1\right) + K^2\left(x^2, y\right)\right) \mid \nabla_{y^2} K^2\left(x^2, y\right) \in C_2^*, \left(x^2, y\right) \in C_1 \times T, \lambda \geq 1 \right\}$$

or $$z_1 = \max_{x^1} \min_{x^2 . y}\left(K^1\left(x^1\right) + f_2(y^1)\right) \tag{3.3}$$

where $$f_2\left(y^1\right) = \operatorname*{Min}_{x^2 . y^2}\left\{K^2\left(x^2, y\right) - \lambda\left(y^2\right) t \nabla_{y^2} K^2\left(x^2, y\right) \mid \nabla_{y^2} K^2\left(x^2, y\right) \in C_2^*, \left(x^2, y^2\right) \in C_1 \times C_2, \lambda \geq 1\right\} \tag{3.4}$$

Similarly, w_1 can be written as $$w_1 = \min_{y^1} \max_{x . y^2}\left(K^1\left(x^1\right) + g_2\left(y^1\right)\right) \tag{3.5}$$

where $$g_2(y^1) = \max_{x^2 . y^2}\left\{K^2\left(x^2, y\right) - \mu\left(x^2\right) t \nabla_{x^2} K^2\left(x^2, y\right) \mid -\nabla_{x^2} K^2\left(x^2, y\right) \in C_1^*, \left(x^2, y^2\right) \in C_1 \times C_2, \mu \geq 1\right\} \tag{3.6}$$

Let $\left(x, y\right) \in P_1$ and $\left(u, v\right) \in D_1$.

It is sufficient to show that $f_2\left(y^1\right) \geq g_2(v_1)$.

Since $x^2 - u^2 \in C_1$ and $-\nabla_{u^2} K^2\left(u^2, v\right) \in C_1^*$,

we have, $-\nabla_{u^2} K^2\left(u^2, v\right)\left(x^2 - u^2\right) \geq 0$ i.e., $\nabla_{u^2} K^2\left(u^2, v\right)\left(x^2 - u^2\right) \geq 0$ (3.7)

Similarly since $v^2 - y^2 \in C_2$ and $\nabla_{y^2} K^2\left(x^2, y\right) \in C_2^*$, we obtain

$$\nabla_{y^2} K^2\left(x^2, y\right)\left(v^2 - y^2\right) \leq 0 \tag{3.8}$$

Since K is increasing/decreasing on $C_1 \times C_2$ by using (3.7) and (3.8)

we have $$K^2\left(x^2, v\right) \geq K^2\left(u^2, v\right) \tag{3.9}$$

and $$K^2\left(x^2, v\right) \leq K^2\left(u^2, v\right) \tag{3.10}$$

From (3.9) and (3.10) it follows that $K^2\left(x^2, y\right) \geq K^2\left(u^2, v\right)$ (3.11)

Since $u^2 \in C_1, -\nabla_{u^2} K^2\left(u^2, v\right) \in C_1^*$ and $\mu \geq 1$, we have

$$\mu\left(u^2\right)^t \nabla_{u^2} K^2\left(u^2, v\right) \leq 0 \tag{3.12}$$

Similarly, since $y^2 \in C_2, \nabla_{y^2} K^2\left(x^2, y\right) \in C_2^*$ and $\lambda \geq 1$, we obtain

$$-\lambda\left(y^2\right) t \nabla_{y^2} K^2\left(x^2, y\right) \geq 0 \tag{3.13}$$

Using (3.12) and (3.13) in (3.11) we have

$$K^2\left(x^2, y\right) - \lambda\left(y^2\right)^t \nabla_{y^2} K^2\left(x^2, y\right) \geq K^2\left(u^2, v\right) + \mu\left(v^2\right)^t \nabla_{u^2} K^2\left(u^2, v\right)$$

$$\Rightarrow \quad f_2\left(y^1\right) \geq g_2\left(v^1\right)$$

This completes the proof.
Before proving the forward duality theorem we state a proposition.

4 Proposition 1

Let X be a convex set with nonempty interior in R^n and C be a reproducing cone in R^m, having nonempty interior. Let f and g be real and vector valued functions respectively defined on X. Consider the problem:

Minimize f(z)
subject to $z = x$, $g(z) \in C$

If z^0 solves the problem then there exists a nonzero (q_0, q) such that $\left(q_0 \nabla f(z_0) + \nabla^t g(z_0) q\right)^t (z - z_0) \geq 0$

for each $z \in X$ and $q_0 \geq 0, q \in C^*$ and $q^t g(z_0) = 0$.

q_0 need not be positive always. However, $q_0 \geq 0$ under suitable constraint qualifications.

Theorem. 2 (Forward Duality)
If K(x, y) is twice differentiable in x^2 and y^2, $\nabla_{y^2y^2} K\left(\bar{x}, \bar{y}\right)$ is negative definite and $\left(\bar{x}, \bar{y}\right)$ is a solution of (P_0) the following statements hold:

(1) $\left(\bar{x}, \bar{y}\right)$ is a solution of (D_0)

(2) $\underset{x^1}{\text{Min}}\ \underset{x^2.y}{\text{Max}} \left\{f \mid \left(x, y\right) \in P_1\right\} = \underset{y^1}{\text{Min}}\ \underset{x.y^2}{\text{Max}} \left\{g \mid \left(x, y\right) \in D_1\right\}$

(3) $\bar{y}_2\ \nabla_{y^2}\ K\left(\bar{x}, \bar{y}\right) = \bar{x}_2\ \nabla_{x^2} K\left(\bar{x}, \bar{y}\right) = 0$

Proof: For a given y^1, (3.4) and (3.6) are a pair nonlinear programs of the type of [28] when $\lambda = \mu = 1$. We follow their approach and use the above proposition with $z = \left(x^2, y^2\right), X = C_1 \times C_2, C = C_2^*$ and

$$f\left(z\right) = K^2\left(x^2, y\right) - \mu\left(y^2\right)^t \nabla_{y^2} K^2\left(x^2, y\right)$$

$$g\left(z\right) = \nabla_{y^2} K^2\left(x^2, y\right)$$

Hence is z_0 solves the problem, there exists a nonzero (q_0, q) such that

$$\left(q_0 \nabla_{x^3}{}^t K^2\left(\bar{x}^2, y^2\right) - q_0 \lambda\left(\bar{y}^2\right)^t \nabla_{y^2x^2} K^2\left(\bar{x}^2, \bar{y}\right) + q^t \nabla_{y^2x^2} K^2\left(\bar{x}^2, \bar{y}\right)\right)\left(x^2 - \bar{x}^2\right)$$

$$+\left(q_0\left(1-\lambda\right)\nabla_{y^2}{}^t K^2\left(\bar{x}^2, \bar{y}\right) + \left(-q_0 \lambda\left(\bar{y}^2\right)^t + q^t\right)\nabla_{y^2y^2} K^2\left(\bar{x}^2, \bar{y}^2\right)\right)\left(y^2 - \bar{y}^2\right) \geq 0 \qquad (3.14)$$

for each $\left(x^2, y^2\right) C_1 \times C_2$ and $q_0 \geq 0, q \in \left(C_2^*\right)^* = C_2$

(Since C_2 is a closed convex cone) and $q^t\nabla_{y^2}K^2\left(x^2, y\right)=0$ (3.15)

We claim that $q_0>0.$

To show this let $x^2=\overline{x}^2$ in (3.14), then we get

$$q_0(1-\lambda)\nabla^t_{y^2}K^2\left(\overline{x}^2,\overline{y}\right)\left(y^2-\overline{y}^2\right)+\left(-\lambda q_0(y^2)^t+q^t\right)\nabla_{y^2y^2}K^2\left(\overline{x}^2,\overline{y}\right)\left(y^2-\overline{y}^2\right)\geq 0 \quad (3.16)$$

for each $y^2\in C_2$. If $q_0=0$ and $y^2=\overline{y}^2+q$, we have from (3.16) $q^t\nabla_{y^2y^2}K^2\left(\overline{x}^2,\overline{y}\right)q\leq 0$ which by negative definiteness of $\nabla_{y^2y^2}K^2\left(\overline{x}^2,\overline{y}^2\right)$ implies that q = 0, but this is impossible since $\left(q_0,q\right)\neq 0$, and therefore $q_0>0$. Let $q=\lambda q_0\overline{y}^2$. From (3.16) we have

$$q_0(1-\lambda)\nabla^t_{y^2}K^2\left(\overline{x}^2,\overline{y}\right)y^2-(1-\lambda)\nabla^t_{y^2}K^2\left(\overline{x}^2,y\right)\frac{q}{\lambda}\geq 0$$

$$\text{or}\quad q_0(1-\lambda)\nabla^t_{y^2}K^2\left(\overline{x}^2,\overline{y}\right)y^2\geq 0$$

by using (3.15), which is always true as $\lambda\geq 1$ and $q_0>0$.

If $q\neq\lambda q_0\overline{y}^2$ it is verified that $y^2=\dfrac{q}{\left(\lambda q_0\right)}\in C_2$ and the relation (3.16) is not valid.

By putting $y^2=\overline{y}^2$ in (3.14) we get $\nabla^t_{x^2}K^2\left(\overline{x}^2,\overline{y}\right)\left(x^2-\overline{x}^2\right)\geq 0$ for each $x^2\in C_1$.

Let $x^2\in C_1$ then $\overline{x}^2+x^2\in C_1$ so that the last inequality implies that

$$\left(x^2\right)^t\nabla_{x^2}K^2\left(\overline{x}^2,\overline{y}\right)\geq 0 \qquad \text{i.e., } -\nabla_{x^2}K^2\left(\overline{x}^2,\overline{y}^2\right)\in C_1^*$$

By setting $x^2=0$ and $x^2=\overline{x}^2$

(1) $(\bar{x}, \bar{y})$ solves (P_0)

(2) $\min_{x^1} \max_{x^2, y} \{f \mid (x, y) \in P_1\} = \min_{y^1} \max_{x, y^2} \{g \mid (x, y) \in D_1\}$

(3) $\bar{x}_2 \nabla_{x^2} K(\bar{x}, \bar{y}) = \bar{y}_2 \nabla_{y^2} K(\bar{x}, \bar{y}) = 0$

The proof of theorem 3 involves arguments similar to that of theorem 2.

5 Conclusion

In this paper we have presented a pair of non-convex mixed integer programming problems which are generally non-symmetric from duality point of view but reduce to a pair of symmetric dual nonlinear mixed integer programs under particular conditions. For this general formulation, we have established the weak, forward and converse duality theorems considering increasing and decreasing functions with an additional feasibility conditions. The results are also given for strong pseudo convex functions.

References

[1] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970. http://dx.doi.org/10.1515/9781400873173

[2] A. M. Geoffrion, Duality in nonlinear programming: a simplified application oriented development, *SIAM Review*, **13** (1971), 1-37. http://dx.doi.org/10.1137/1013001

[3] J. Stoer, C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, Springer Verlag, New York, 1970. http://dx.doi.org/10.1007/978-3-642-46216-0

[4] G. B. Dantzig, E. Eisenberg, R. W. Cottle, Symmetric dual nonlinear programs, *Pacific Journal of Mathematics*, **15** (1965), no. 3, 809-812. http://dx.doi.org/10.2140/pjm.1965.15.809

[5] E. Balas, Minimax and duality for linear and nonlinear mixed-integer Programming, J. Abadie (ed), *Integer and Nonlinear Programming*, North Holand, Amsterdam, 1970.

[6] B. K. Mishra, C. Das, Minimax and symmetric duality for a nonlinear mixed Integer programming problem, *Opsearch*, **17** (1980), no. 1, 1-11.

[7] M. S. Mishra, D. Acharya, S. Nanda, On a pair of nonlinear mixed integer programming problems, *European Journal of Operations Research*, **19** (1985), 98-103. http://dx.doi.org/10.1016/0377-2217(85)90313-3

[8] W. S. Dorn, A symmetric dual theorem for quadratic programming, *Journal of the Operations Research Society of Japan*, **2** (1960), 93-97.

[9] V. Kumar, I. Hussain, S. Chandra, Symmetric duality for minimax nonlinear mixed integer programming, *European Journal of Operational Research*, **80** (1995), 425-430. http://dx.doi.org/10.1016/0377-2217(93)e0293-7

[10] B. Mond, A symmetric dual theorem for nonlinear programs, *Quarterly Journal of Applied Mathematics*, **23** (1965), 265-269.

[11] B. Mond, M. A. Hanson, On duality with generalized convexity, *Mathematische Operationsforschung und Statistik. Series Optimization*, **15** (1984), no. 2, 313-317. http://dx.doi.org/10.1080/02331938408842939

[12] B. Mond, T. Weir, Generalized concavity and duality in S. Schaible, W. T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981, 263-279.

[13] J. R. Nayak, Some Problems of Non-Convex Programming and the Properties of Some Non-convex Functions, Ph. D. Thesis, Utkal University, 2004.

[14] S. Chandra, B. Mond, I. Smart, Constrained games and symmetric duality with pseudo- invexity, *Opsearch*, **27** (1990), 114-30.

[15] I. Ahmad, Z. Husain, Minimax mixed integer symmetric duality for multiobjective variational problems, *European Journal of Operational Research*, **177** (2007), 71-82. http://dx.doi.org/10.1016/j.ejor.2005.06.070

[16] Ning Ruan, Complete Solutions to Mixed Integer Programming, *American Journal of Computational Mathematics*, **3** (2013), no. 3, 27-30. http://dx.doi.org/10.4236/ajcm.2013.33b005

[17] B. Mond, S. Chandra and I. Hussain, Duality for Variational Problems with Invexity, *Journal of Mathematical Analysis and Application*, **134** (1988), 322-328. http://dx.doi.org/10.1016/0022-247x(88)90026-1

[18] B. K. S. Cheung, A. Langevin, H. Delmaire, Coupling genetic algorithm with a grid search method to solve mixed integer nonlinear programming problems, *Computers and Mathematics with Applications*, **32** (1997), no. 12, 13-23. http://dx.doi.org/10.1016/s0898-1221(97)00229-0

[19] M. F Cardoso, R.L. Salcedo, V. Feyo de Azevedo, V. Barbosa, A Simulated annealing approach to the solution of minlp problems, *Comput. Chem. Eng.*, **21** (1997), no.12, 1349–1364. http://dx.doi.org/10.1016/s0098-1354(97)00015-x

[20] M. A Duran, I.E Grossmann, A mixed-integer nonlinear programming approach for process synthesis, *AIChE. Journal*, **32** (1986), no. 4, 592-606. http://dx.doi.org/10.1002/aic.690320408

[21] C. A. Floudas, A. Aggarwal, A.R. Ciric, Global optimum search for nonconvex NLP and MINLP programs, *Comput. Chem. Eng.*, **13** (1989), no. 10, 1117–1132. http://dx.doi.org/10.1016/0098-1354(89)87016-4

[22] C. A. Floudas, *Nonlinear and Mixed-Integer Optimization Fundamentals and Applications*, Oxford University Press, Oxford, 1995.

[23] I. E. Grossmann, Z. Kravanja, Mixed-integer nonlinear programming techniques for process systems engineering, *Comput. Chem. Eng.*, **19** (1995), 189–204. http://dx.doi.org/10.1016/0098-1354(95)87036-9

[24] H. S. Ryoo, N.V.Sahinidis, Global optimization of nonconvex NLPs and MINLPs with applications in process design, *Comput. Chem. Eng.*, **19** (1995), no. 5, 551–566. http://dx.doi.org/10.1016/0098-1354(94)00097-2

[25] R. L. Salcedo, Solving nonconvex nonlinear programming and mixed-integer non-linear programming problems with adaptive random search, *Ind. Eng. Chem. Res.*, **31** (1992), 262–273. http://dx.doi.org/10.1021/ie00001a037

[26] X. Yuan, Mixed integer nonlinear programming and optimal design of chemical engineering system (I) A MINLP algorithm for engineering system design, *J. Chem. Ind. Eng. (China)*, **42** (1991), no. 1, 33–39.

[27] G. R. Kocis, I.E. Grossmann, Global optimization of nonconvex mixed integer nonlinear programming (MINLP) problems in process synthesis, *Ind. Eng. Chem. Res.*, **27** (1988), 1407-1421. http://dx.doi.org/10.1021/ie00080a013

[28] M. S. Bazaraa, J. J. Goode, On Symmetric Duality in Non Linear Programming, *Operations Research*, **21** (1973), no. 1, 1-9. http://dx.doi.org/10.1287/opre.21.1.1

[29] S. Chandra, B.D. Caven, B. Mond, Symmetric dual fractional programming, *Zeitschrift for Operations Research*, **29** (1985), 59-64. http://dx.doi.org/10.1007/bf01920497

Received: October 5, 2015; Published: December 2, 2015