Mixed Means Inequalities of Multivariable Geometric Mean and Harmonic Mean

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Abstract

In this paper, we establish several mixed inequalities of multivariable geometric mean and harmonic mean by the theory of Schur convexity and majorization.

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1 Introduction

Throughout this paper we denote $R^n (n \geq 2)$ the $n$-dimensional Euclidean space, $R_+^n = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$ and $R = R^1$.

For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in R_+^n$ and $\alpha > 0$, we denote by

\[
\begin{align*}
x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), \\
x y &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\
\alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\
x^\alpha &= (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \\
\frac{1}{x} &= \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right), \\
\log x &= (\log x_1, \log x_2, \cdots, \log x_n)
\end{align*}
\]

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and
\[ e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n}), \]

\[ A_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad G_n(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \quad \text{and} \quad H_n(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \]

denote the un-weighted arithmetic, geometric and harmonic means of \( x \), respectively.

For \( x = (x_1, x_2, \cdot, x_n) \in R^n_+ \) and \( w \geq 0 \), the mixed means \( H_w(x) \) of arithmetic mean and geometric mean are defined by K.Z. Guan and H.T. Zhu [5] as follows:

\[
H_w(x) = H_w(x_1, x_2, \cdots, x_n) = \begin{cases} 
\frac{nA_n(x) + wG_n(x)}{w+n}, & 0 \leq w < +\infty, \\
G_n(x), & w = +\infty.
\end{cases}
\]

In [5], K.Z. Guan and H.T. Zhu proved that \( H_w(x) \) is Schur concave in \( R^n_+ \) for any \( w > 0 \), and established several ratio inequalities and Ky Fan type inequalities involving the mean \( H_w(x) \).

For \( x = (x_1, x_2, \cdot, x_n) \in R^n_+ \) and \( w \geq 0 \), in this paper we define the mixed means of multivariable geometric mean and harmonic mean as follows:

\[
M_w(x) = M_w(x_1, x_2, \cdots, x_n) = \begin{cases} 
\frac{H_n(x) + wG_n(x)}{1+w}, & 0 \leq w < +\infty, \\
G_n(x), & w = +\infty.
\end{cases}
\]

The main purpose of this paper is to establish some inequalities for the mean \( M_w(x) \).

2 Preliminary knowledge

For the sake of readability, in this section we introduce some definitions and well-known results as follows.

**Definition 2.1.** Let \( E \subseteq R^n \) be a set, a real-valued function \( F \) on \( E \) is called a Schur convex function if

\[
F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)
\]

for each pair of \( n \)-tuples \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \) in \( E \), such that \( x \prec y \), i.e.

\[
\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1
\]
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and

\[ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \]

where \( x_{[i]} \) denotes the \( i \)th largest component of \( x \). A function \( F \) is called Schur concave if \( -F \) is Schur convex.

**Definition 2.2.** Let \( E \subseteq R_{+}^{n} \) be a set, \( F : E \rightarrow R_{+} \) is called Schur multiplicatively convex on \( E \) if \( F(x_{1}, x_{2}, \cdots, x_{n}) \leq F(y_{1}, y_{2}, \cdots, y_{n}) \) for each pair of \( n \)-tuples \( x = (x_{1}, x_{2}, \cdots, x_{n}) \) and \( y = (y_{1}, y_{2}, \cdots, y_{n}) \) in \( E \), such that \( \log x \prec \log y \). \( F \) is called Schur multiplicatively concave if \( \frac{1}{F} \) is Schur multiplicatively convex.

**Definition 2.3.** Let \( E \subseteq R_{+}^{n} \) be a set, \( F : E \rightarrow R_{+} \) is called Schur harmonic convex (or Schur harmonic concave, respectively) on \( E \) if

\[ F(x_{1}, x_{2}, \cdots, x_{n}) \leq (\text{or} \geq, \text{respectively}) F(y_{1}, y_{2}, \cdots, y_{n}) \]

for each pair of \( n \)-tuples \( x = (x_{1}, x_{2}, \cdots, x_{n}) \) and \( y = (y_{1}, y_{2}, \cdots, y_{n}) \) in \( E \), such that \( \frac{1}{x} \prec \frac{1}{y} \).

Definitions 2.1, 2.2, and 2.3 have the following consequences.

**Remark 2.1.** Let \( E \subseteq R_{+}^{n} \) be a set, and \( H = \log E = \{ \log x : x \in E \} \). Then \( f : E \rightarrow R_{+} \) is Schur multiplicatively convex (or Schur multiplicatively concave, respectively) on \( E \) if and only if \( \log f(e^{x}) \) is Schur concave (or Schur convex, respectively) on \( H \).

**Remark 2.2.** Let \( E \subseteq R_{+}^{n} \) be a set, and \( H = \frac{1}{E} = \{ \frac{1}{x} : x \in E \} \). Then \( f : E \rightarrow R_{+} \) is Schur harmonic convex (or Schur harmonic concave, respectively) on \( E \) if and only if \( \frac{1}{f(x)} \) is Schur concave (or Schur convex, respectively) on \( H \).

Schur convexity was introduced by I. Schur [9] in 1923, it has many applications in inequality theory [1, 6, 13]. Recently, the Schur multiplicative convexity was investigated in [2, 3, 7], but no one has ever researched the Schur harmonic convexity.

The following well-known result was proved by A.W. Marshall and I. Olkin [8].

**Theorem A.** Let \( E \subseteq R_{+}^{n} \) be a symmetric convex set with nonempty interior \( \text{int} E \) and \( f : E \rightarrow R \) be a continuous symmetric function. If \( f \) is differentiable on \( \text{int} E \), then \( f \) is Schur convex on \( E \) if and only if

\[ (x_{i} - x_{j})(\frac{\partial f}{\partial x_{i}} - \frac{\partial f}{\partial x_{j}}) \geq 0 \quad (2.1) \]

for all \( i, j = 1, 2, \cdots, n \) and \( x = (x_{1}, x_{2}, \cdots, x_{n}) \in \text{int} E \). Here \( E \) is a symmetric set means that \( x \in E \) implies \( Px \in E \) for any \( n \times n \) permutation matrix \( P \).
Remark 2.3. Since \( f \) is symmetric, the Schur’s condition in Theorem A, i.e. (2.1) can be reduced as
\[
(x_1 - x_2)(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}) \geq 0.
\]

The following Theorems B and C can be derived from Remarks 2.1-2.3 and Theorem A.

**Theorem B\([2]\).** Let \( E \subseteq R^n_+ \) be a symmetric multiplicatively convex set with nonempty interior \( \text{int} E \) and \( f : E \to R_+ \) be a continuous symmetric function. If \( f \) is differentiable on \( \text{int} E \), then \( f \) is Schur multiplicatively convex on \( E \) if and only if
\[
(\log x_1 - \log x_2)(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}) \geq 0
\]
for all \((x_1, x_2, \cdots, x_n) \in \text{int} E\). Here \( E \subseteq R^n_+ \) is a multiplicatively convex set means that \( x_1^\frac{1}{2} y^\frac{1}{2} \in E \) whenever \( x, y \in E \).

**Theorem C.** Let \( E \subseteq R^n_+ \) be a symmetric harmonic convex set with nonempty interior \( \text{int} E \) and \( f : E \to R_+ \) be a continuous symmetric function. If \( f \) is differentiable on \( \text{int} E \), then \( f \) is Schur harmonic convex on \( E \) if and only if
\[
(x_1 - x_2)(x_1^2 \frac{\partial f}{\partial x_1} - x_2^2 \frac{\partial f}{\partial x_2}) \geq 0
\]
for all \((x_1, x_2, \cdots, x_n) \in \text{int} E\). Here \( E \subseteq R^n_+ \) is a harmonic convex set means that \( \frac{x y}{x + y} \in E \) whenever \( x, y \in E \).

## 3 Lemmas

In this section, we establish several lemmas which are crucial in the proof of our main results in next section.

**Lemma 3.1.** If \( w \geq 0 \), then \( M_w(x) \) is
(i) Schur concave in \( R^n_+ \);
(ii) Schur multiplicatively concave in \( R^n_+ \);
(iii) Schur harmonic convex in \( R^n_+ \).

**Proof.** If \( w = +\infty \), then (1.1) leads to that
\[
\frac{\partial M_w(x)}{\partial x_i} = \frac{G_w(x)}{nx_i}, i = 1, 2, \cdots, n,
\]
(3.1)
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\[(x_1 - x_2) \left( \frac{\partial M_w(x)}{\partial x_1} - \frac{\partial M_w(x)}{\partial x_2} \right) = -\frac{(x_1 - x_2)^2 G_n(x)}{nx_1x_2} \leq 0, \quad (3.2)\]

\[(\log x_1 - \log x_2) \left( x_1 \frac{\partial M_w(x)}{\partial x_1} - x_2 \frac{\partial M_w(x)}{\partial x_2} \right) = 0 \quad (3.3)\]

and

\[(x_1 - x_2) \left( x_1^2 \frac{\partial M_w(x)}{\partial x_1} - x_2^2 \frac{\partial M_w(x)}{\partial x_2} \right) = \frac{(x_1 - x_2)^2 G_n(x)}{n} \geq 0. \quad (3.4)\]

If \(0 \leq w < +\infty\), then (1.1) implies that

\[\frac{\partial M_w(x)}{\partial x_i} = \frac{H^2_n(x)}{n(w + 1)x_i^2} + \frac{wG_n(x)}{n(w + 1)x_i}, \quad (3.5)\]

\[(x_1 - x_2) \left( \frac{\partial M_w(x)}{\partial x_1} - \frac{\partial M_w(x)}{\partial x_2} \right)
= -\frac{(x_1 - x_2)^2}{n(w + 1)x_1x_2} \left[ (x_1 + x_2)H^2_n(x) + wx_1x_2G_n(x) \right] \leq 0, \quad (3.6)\]

\[(\log x_1 - \log x_2) \left( x_1 \frac{\partial M_w(x)}{\partial x_1} - x_2 \frac{\partial M_w(x)}{\partial x_2} \right)
= -\frac{(x_1 - x_2)(\log x_1 - \log x_2)}{n(w + 1)x_1x_2} H^2_n(x) \leq 0 \quad (3.7)\]

and

\[(x_1 - x_2) \left( x_1^2 \frac{\partial M_w(x)}{\partial x_1} - x_2^2 \frac{\partial M_w(x)}{\partial x_2} \right)
= \frac{w}{n(w + 1)}(x_1 - x_2)^2 G_n(x) \geq 0. \quad (3.8)\]

Therefore, Lemma 3.1 (i) follows from (3.2), (3.6), Theorem A and Remark 2.1 together with Definition 2.1; Lemma 3.1 (ii) follows from (3.3), (3.7), Theorem B and Definition 2.2; Lemma 3.1 (iii) follows from (3.4), (3.8), Theorem C and Definition 2.3.

**Lemma 3.2.** If \(w \geq 1\), then the function \(\Phi_w(x) = \frac{M_w(x)}{M_{w-1}(x)}\) is

(i) Schur multiplicatively convex in \(R^n_+\);

(ii) Schur harmonic convex in \(R^n_+\).
Proof. If \( w = +\infty \), then Lemma 3.2 is trivial. If \( 1 \leq w < +\infty \), then (1.1) leads to that
\[
\begin{align*}
& (\log x_1 - \log x_2) \left( x_1 \frac{\partial \Phi_w(x)}{\partial x_1} - x_2 \frac{\partial \Phi_w(x)}{\partial x_2} \right) \\
& = \frac{w(x_1 - x_2)(\log x_1 - \log x_2)}{n(w + 1)x_1x_2[H_n(x) + (w - 1)G_n(x)]^2} G_n(x)H_n^2(x) \\
& \geq 0
\end{align*}
\]
and
\[
\begin{align*}
& (x_1 - x_2) \left( x_1 \frac{\partial \Phi_w(x)}{\partial x_1} - x_2 \frac{\partial \Phi_w(x)}{\partial x_2} \right) \\
& = \frac{w(x_1 - x_2)^2}{n(w + 1)[H_n(x) + (w - 1)G_n(x)]^2} H_n(x)G_n(x) \\
& \geq 0.
\end{align*}
\]

Therefore, Lemma 3.2 (i) follows from (3.9) and Theorem B, and Lemma 3.2 (ii) follows from (3.10) and Theorem C.

1.3 Remark 3.1. According to (1.1), Theorem A and Definition 2.1, it is not difficult to verify that \( \frac{M_w(x)}{M_{w-1}(x)} \) is Schur convex in \( R^2_+ \) for all \( w \geq 1 \), and \( \frac{M_w(x)}{M_{w-1}(x)} \) is neither Schur convex nor Schur concave in \( R^n_+ \) for any \( w \geq 1 \) and \( n \geq 3 \).

### 4 Main results

1.3 Theorem 4.1. If \( w \geq 0 \), then

(i) \( M_{w+1}(x)M_{w+\alpha}(x) \geq M_w(x)M_{w+1+\alpha}(x) \) for all \( x \in R^n_+ \) and \( \alpha \geq 0 \);

(ii) \( \frac{H_n(x)}{H_{n(1-x)}} \leq \frac{M_w(x)}{M_{w-1}(x)} \leq \frac{G_n(x)}{G_{n(1-x)}} \) for \( x \in (0, \frac{1}{2}]^n \).

Proof. (i) By (1.1) we have
\[
\frac{d}{dw} \left( \frac{M_{w+1}(x)}{M_w(x)} \right) = \frac{d}{dw} \left( \frac{w + 1}{w + 2} \frac{H_n(x) + (w + 1)G_n(x)}{H_n(x) + wG_n(x)} \right)
\]
\[
= \frac{[H_n(x) + 2(w + 1)G_n(x)][H_n(x) - G_n(x)]}{(w + 2)^2[H_n(x) + wG_n(x)]} \\
\leq 0.
\]

Inequality (4.1) implies that \( \frac{M_{w+1}(x)}{M_w(x)} \) is decreasing with respect to \( w \) for any fixed \( x \in R^n_+ \), this leads to Theorem 4.1(i).
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(iii) A result due to W.L. Wang and P.F. Wang[10] gives

\[
\frac{H_n(x)}{H_n(1-x)} \leq \frac{G_n(x)}{G_n(1-x)}
\]  

(4.2)

for \( x = (x_1, x_2, \cdots, x_n) \in (0, \frac{1}{2}]^n \).

If \( w = +\infty \), then from (1.1) and (4.2) we clearly see that Theorem 4.1(ii) is true.

If \( 0 \leq w < +\infty \), then (1.1) leads to that

\[
\frac{M_w(x)}{M_w(1-x)} = \frac{wG_n(x) + H_n(x)}{wG_n(1-x) + H_n(1-x)}
\]

and

\[
\frac{d}{dw} \left( \frac{M_w(x)}{M_w(1-x)} \right) = \frac{G_n(x)H_n(1-x) - H_n(x)G_n(1-x)}{[wG_n(1-x) + H_n(1-x)]^2}.
\]

(4.3)

Inequalities (4.2) and (4.3) imply that \( \frac{M_w(x)}{M_w(1-x)} \) is increasing with respect to \( w \geq 0 \) for any fixed \( x \in (0, \frac{1}{2}]^n \). From this monotonicity and the fact that \( M_0(x) = H_n(x) \) and \( M_{+\infty}(x) = G_n(x) \), we clearly see that Theorem 4.1(ii) is true.

1.5 Theorem 4.2. Suppose that \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n \), and \( \sum_{i=1}^{n} x_i = s \). If \( c \geq s \) and \( w \geq 0 \), then

(i) \( M_w(x) \leq \frac{1}{s-1} M_w(c-x) = \frac{1}{s-1} M_w(c-x_1, c-x_2, \cdots, c-x_n) \);

(ii) \( M_w\left(\frac{1}{c-x}\right) \geq (\frac{w}{s}-1) M_w\left(\frac{1}{c-x_1}, \frac{1}{c-x_2}, \cdots, \frac{1}{c-x_n}\right) \);

(iii) \( \frac{M_{w+1}\left(\frac{1}{c-x}\right)}{M_w\left(\frac{1}{c-x}\right)} \geq \frac{M_{w+1}\left(\frac{1}{c-x_1}, \frac{1}{c-x_2}, \cdots, \frac{1}{c-x_n}\right)}{M_w\left(\frac{1}{c-x_1}, \frac{1}{c-x_2}, \cdots, \frac{1}{c-x_n}\right)} \).

Proof. A result from [4, Lemma 2.3] gives

\[
\frac{c-x}{w} = \left( \frac{c-x_1}{w} \frac{c-x_2}{w} \cdots \frac{c-x_n}{w} \right) < (x_1, x_2, \cdots, x_n) = x.
\]

(4.4)

Therefore, Theorem 4.2(i) follows from Lemma 3.1(i), (4.4) and (1.1); Theorem 4.2(ii) follows from Lemma 3.1(iii), (4.4) and (1.1); Theorem 4.2(iii) follows from Lemma 3.2(ii), (4.4) and (1.1).

1.5 Theorem 4.3. Suppose that \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n \), and \( \sum_{i=1}^{n} x_i = s \). If \( c \geq 0 \) and \( w \geq 0 \), then

(i) \( M_w(x) \leq \frac{1}{w+1} M_w(c+x) = \frac{1}{w+1} M_w(c+x_1, c+x_2, \cdots, c+x_n) \);

(ii) \( M_w\left(\frac{1}{c+x}\right) \geq (\frac{w}{s}+1) M_w\left(\frac{1}{c+x_1}, \frac{1}{c+x_2}, \cdots, \frac{1}{c+x_n}\right) \);

(iii) \( \frac{M_{w+1}\left(\frac{1}{c+x}\right)}{M_w\left(\frac{1}{c+x}\right)} \geq \frac{M_{w+1}\left(\frac{1}{c+x_1}, \frac{1}{c+x_2}, \cdots, \frac{1}{c+x_n}\right)}{M_w\left(\frac{1}{c+x_1}, \frac{1}{c+x_2}, \cdots, \frac{1}{c+x_n}\right)} \).
Proof. A result from [4, Lemma 2.4] gives
\[
\frac{c + x}{\frac{nc}{s} + 1} = \left( \frac{c + x_1}{\frac{nc}{s} + 1}, \frac{c + x_2}{\frac{nc}{s} + 1}, \ldots, \frac{c + x_n}{\frac{nc}{s} + 1} \right) \succ (x_1, x_2, \ldots, x_n) = x. \tag{4.5}
\]

Therefore, Theorem 4.3(i) follows from Lemma 3.1(i), (4.5) and (1.1); Theorem 4.3(ii) follows from Lemma 3.1(iii), (4.5) and (1.1); Theorem 4.3(iii) follows from Lemma 3.2(ii), (4.5) and (1.1).

1.5 Theorem 4.4. Suppose that \( x = (x_1, x_2, \ldots, x_n) \in R^n_+ \) and \( \sum_{i=1}^n x_i = s \).

If \( 0 \leq \lambda \leq 1 \) and \( w \geq 0 \), then
(i) \( M_w(x) \leq \frac{1}{n-\lambda} M_w(s - \lambda x) \);
(ii) \( M_w(x) \leq \frac{1}{n+\lambda} M_w(s + \lambda x) \);
(iii) \( M_w(\frac{1}{x}) \geq (n - \lambda) M_w(\frac{1}{s + \lambda x}) \);
(iv) \( M_w(\frac{1}{x}) \geq (n + \lambda) M_w(\frac{1}{s - \lambda x}) \);
(v) \( \frac{M_{w+1}(\frac{1}{x})}{M_w(\frac{1}{x})} \geq \frac{M_{w+1}(\frac{1}{s + \lambda x})}{M_w(\frac{1}{s + \lambda x})} \);
(vi) \( \frac{M_{w+1}(\frac{1}{x})}{M_w(\frac{1}{x})} \geq \frac{M_{w+1}(\frac{1}{s - \lambda x})}{M_w(\frac{1}{s - \lambda x})} \).

Proof. A result due to S.H. Wu [12, Lemma 2] gives
\[
\frac{s - \lambda x}{n - \lambda} = \left( \frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \ldots, \frac{s - \lambda x_n}{n - \lambda} \right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{4.6}
\]

It is not difficult to verify that
\[
\frac{s + \lambda x}{n + \lambda} = \left( \frac{s + \lambda x_1}{n + \lambda}, \frac{s + \lambda x_2}{n + \lambda}, \ldots, \frac{s + \lambda x_n}{n + \lambda} \right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{4.7}
\]

Therefore, Theorem 4.4 follows from Lemma 3.1(i) and (iii), Lemma 3.2(ii) and (1.1) together with (4.6) and (4.7).

Theorem 4.5. Suppose that \( A = A_1 A_2 \cdots A_{n+1} \) be an \( n \)-dimensional simplex in \( R^n \) (\( n \geq 3 \)). Let \( P \) be an arbitrary point in the interior of \( A \), and \( B_i \) stand for the intersection point of straight line \( A_i P \) and the hyperplane \( \Sigma_i = A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_n A_{n+1}, i = 1, 2, \ldots, n, n+1 \). If \( w \geq 0 \), then 1.5
(i) \( M_w\left( \frac{P B_1}{A_1 B_1}, \frac{P B_2}{A_2 B_2}, \ldots, \frac{P B_{n+1}}{A_{n+1} B_{n+1}} \right) \leq \frac{1}{n+1} \);
(ii) \( M_w\left( \frac{A_1 P}{A_1 B_1}, \frac{A_2 P}{A_2 B_2}, \ldots, \frac{A_{n+1} P}{A_{n+1} B_{n+1}} \right) \leq \frac{n}{n+1} \);
(iii) \( M_w\left( \frac{A_1 B_1}{P B_1}, \frac{A_2 B_2}{P B_2}, \ldots, \frac{A_{n+1} B_{n+1}}{P B_{n+1}} \right) \geq n + 1 \);
(iv) \( M_w\left( \frac{A_1 B_1}{A_1 P}, \frac{A_2 B_2}{A_2 P}, \ldots, \frac{A_{n+1} B_{n+1}}{A_{n+1} P} \right) \geq \frac{n+1}{n} \).
Proof. One can easily see that $\sum_{i=1}^{n+1} \frac{P_{B_i}}{\lambda_i} = 1$ and $\sum_{i=1}^{n+1} \frac{A_i P}{\lambda_i B_i} = n$. Therefore, Theorem 4.5 follows from Lemma 3.1(i) and (iii), (1.1) together with the fact that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1 B_1}, \frac{PB_2}{A_2 B_2}, \ldots, \frac{PB_{n+1}}{A_{n+1} B_{n+1}}\right)$$

and

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \ldots, \frac{n}{n+1}\right) \prec \left(\frac{A_1 P}{A_1 B_1}, \frac{A_2 P}{A_2 B_2}, \ldots, \frac{A_{n+1} P}{A_{n+1} B_{n+1}}\right).$$

**Theorem 4.6.** Suppose that $A \in M_n(C)$ ($n \geq 2$) is a complex matrix, $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the eigenvalues and singular values of $A$, respectively. If $A$ is a positive definite Hermitian matrix and $w \geq 0$, then

(i) $M_w(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq \frac{\text{tr} A}{n}$;

(ii) $M_w(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}) \geq \frac{\text{tr} A}{n}$;

(iii) $M_w(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq \sqrt[2]{\det A}$;

(iv) $M_w(1 + \lambda_1, 1 + \lambda_2, \ldots, 1 + \lambda_n) \leq \sqrt[2]{\det(I + A)}$;

(v) $M_w(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq M_w(\lambda_1, \lambda_2, \ldots, \lambda_n)$;

(vi) $M_w(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}) \leq \frac{M_{w+1}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n})}{M_w(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_n})}$.

**Proof.** We clearly see that $\lambda_i > 0$, $\sigma_i > 0$ ($i = 1, 2, \ldots, n$), $\sum_{i=1}^{n} \lambda_i = \text{tr} A$, $\prod_{i=1}^{n} \lambda_i = \det A$ and $\prod_{i=1}^{n} (1 + \lambda_i) = \det(I + A)$. These lead to that

$$\left(\frac{\text{tr} A}{n}, \frac{\text{tr} A}{n}, \ldots, \frac{\text{tr} A}{n}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

$$\log\left(\sqrt[n]{\det A}, \sqrt[n]{\det A}, \ldots, \sqrt[n]{\det A}\right) \prec \log(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and

$$\log(\sqrt[2]{\det(I + A)}, \sqrt[2]{\det(I + A)}, \ldots, \sqrt[2]{\det(I + A)}) \prec \log(1 + \lambda_1, 1 + \lambda_2, \ldots, 1 + \lambda_n).$$

(4.11)

A result due to H. Weyl [11] gives

$$\log(\lambda_1, \lambda_2, \ldots, \lambda_n) \prec \log(\sigma_1, \sigma_2, \ldots, \sigma_n).$$

(4.12)

Therefore, Theorem 4.6 follows from Lemma 3.1, 3.2, and (1.1) together with (4.8)-(4.11).
References


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