New Properties for Starlike and Convex Functions of Complex Order

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Abstract

For functions \( f(z) \) which are starlike of complex order \( b \) \((b \neq 0)\) in the open unit disk \( U \) introduced by M. A. Nasr and M. K. Aouf (J. Natural Sci. Math. 25(1985), 1 - 12), in consideration of its properties, some sufficient conditions, necessary conditions and distortion theorems with coefficient inequalities of \( f(z) \) as the improvement of the well-known result due to H. Silverman (Proc. Amer. Math. Soc. 51(1975), 109 - 116) are discussed. Furthermore, some new coefficient inequalities including some new interesting corollaries are discussed.

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1 Introduction and Preliminaries

Let \( A \) be the class of functions \( f(z) \) of the form

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_0 = 0, \quad a_1 = 1)
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \).

Furthermore, let \( P \) denote the class of functions \( p(z) \) of the form

\[
(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]
which are analytic in \( U \). If \( p(z) \in P \) satisfies \( \Re \, p(z) > 0 \) \((z \in \mathbb{U})\), then we say that \( p(z) \) is the Carathéodory function (cf. [1]).

If \( f(z) \in A \) satisfies the following inequality
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})
\]
for some \( \alpha \) \((0 \leq \alpha < 1)\), then \( f(z) \) is said to be starlike of order \( \alpha \) in \( \mathbb{U} \). We denote by \( S^*(\alpha) \) the subclass of \( A \) consisting of functions \( f(z) \) which are starlike of order \( \alpha \) in \( \mathbb{U} \). Similarly, we say that \( f(z) \) is a member of the class \( K(\alpha) \) of convex functions of order \( \alpha \) in \( \mathbb{U} \) if \( f(z) \in A \) satisfies the following inequality
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U})
\]
for some \( \alpha \) \((0 \leq \alpha < 1)\).

As usual, in the present investigation, we write
\[
S^* = S^*(0) \quad \text{and} \quad K = K(0).
\]
Classes \( S^*(\alpha) \) and \( K(\alpha) \) were introduced by Robertson [5].

Next, a function \( f(z) \in A \) is called \( \lambda \)-spiral like of order \( \alpha \) in \( \mathbb{U} \) if and only if
\[
\Re \left[ e^{i\lambda} \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right] > 0 \quad (z \in \mathbb{U})
\]
for some real \( \lambda \) \((-\frac{\pi}{2} < \lambda < \frac{\pi}{2})\) and \( \alpha \) \((0 \leq \alpha < 1)\). We denote this class by \( S\mathcal{P}(\lambda, \alpha) \).

Moreover, for some non-zero complex number \( b \), we consider the subclasses \( S^*_b \) and \( K_b \) of \( A \) as follows:
\[
S^*_b = \left\{ f(z) \in A : \Re \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > 0 \quad (z \in \mathbb{U}) \right\}
\]
and
\[
K_b = \left\{ f(z) \in A : \Re \left[ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right] > 0 \quad (z \in \mathbb{U}) \right\}.
\]
If a function \( f(z) \) belongs to the class \( S^*_b \) or \( K_b \), we say that \( f(z) \) is starlike or convex of complex order \( b \) \((b \neq 0)\), respectively. In [3], Nasr and Aouf introduced the class \( S^*_b \).
Then, we can see that
\[ S^{*}_{1-\alpha} = S^{*}(\alpha), \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha) \quad \text{and} \quad S^{*}_{(1-\alpha)e^{-i\lambda}} = \mathcal{S}\mathcal{P}(\lambda, \alpha). \]

\textbf{Example 1.1}

\begin{equation}
(1.3) \quad f_b(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n} (j + 2(b - 1))}{(n-1)!} z^n \in S^*_b \quad (b \neq 0)
\end{equation}

and

\begin{equation}
(1.4) \quad g_b(z) = \begin{cases} 
1 - (1-z)^{1-2b} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n} (j + 2(b - 1))}{n!} z^n \in \mathcal{K}_b & (b \neq \frac{1}{2}) \\
\log \left( \frac{1}{1-z} \right) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n \in \mathcal{K}_{\frac{1}{2}} = \mathcal{K}(\frac{1}{2}). 
\end{cases}
\end{equation}

\textbf{Theorem 1.2} \quad \text{For a function } f(z) \in \mathcal{A}, \text{ it follows that}

\[ f(z) \in \mathcal{K}_b \quad \text{if and only if} \quad z\{f'(z)\}^{\frac{1}{b}} \in \mathcal{S}^* \]

\text{and}

\[ f(z) \in \mathcal{S}^* \quad \text{if and only if} \quad \int_{0}^{z} \left( \frac{f(\xi)}{\xi} \right)^{b} d\xi \in \mathcal{K}_b. \]

\textbf{Proof.} \quad \text{We set } F(z) = z\{f'(z)\}^{\frac{1}{b}}. \text{ If } f(z) \in \mathcal{K}_b, \text{ then}

\[ \text{Re} \left( \frac{zF'(z)}{F(z)} \right) = \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right] > 0. \]

Therefore, \( F(z) = z\{f'(z)\}^{\frac{1}{b}} \in \mathcal{S}^* \). The converse is also proved. \hfill \square

Setting \( b = 1 - \alpha \) in Theorem 1.2, we have

\textbf{Corollary 1.3} \quad \text{For a function } f(z) \in \mathcal{A}, \text{ it follows that}

\[ f(z) \in \mathcal{K}(\alpha) \quad \text{if and only if} \quad z\{f'(z)\}^{\frac{1}{1-\alpha}} \in \mathcal{S}^* \]
and
\[ f(z) \in S^* \text{ if and only if } \int_0^z \left( \frac{f(\xi)}{\xi} \right)^{1-\alpha} d\xi \in K(\alpha). \]

Silverman [6] has proved the following well-known theorem.

**Theorem 1.4** If a function \( f(z) \in A \) satisfies the following inequality
\[
\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha \quad \text{or} \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq 1 - \alpha
\]
for some \( \alpha \) (\( 0 \leq \alpha < 1 \)), then \( f(z) \in S^*(\alpha) \) or \( f(z) \in K(\alpha) \), respectively.

We apply the following lemma (see, [7]) to obtain our results.

**Lemma 1.5** A function \( p(z) \in P \) satisfies \( \text{Re} \, p(z) > 0 \) (\( z \in U \)) if and only if
\[
p(z) \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in U)
\]
for all \( |\zeta| = 1 \).

Then, by using Lemma 1.5, various conditions for starlike functions are studied. The following results are enumerated as the some examples.

Silverman, Silvia and Telage [7] have given

**Theorem 1.6** A function \( f(z) \in A \) is in \( S^*(\alpha) \) if and only if
\[
\frac{1}{z} \left( f(z) * \frac{z + \frac{\zeta + 2\alpha - 1}{2} z^2}{\frac{2 - 2\alpha}{(1 - z)^2}} \right) \neq 0 \quad (z \in U, \ |\zeta| = 1)
\]
where \( * \) means the convolution or Hadamard product of two functions.

Furthermore, letting \( \alpha = 0 \) in Theorem 1.6, Nezhmetdinov and Ponnusamy [4] have given the sufficient conditions for coefficients of \( f(z) \) to be in the class \( S^* \).
Hayami, Owa and Srivastava [2] have shown the following results.

**Theorem 1.7** If \( f(z) \in A \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j + 1 - 2\alpha)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \\
+ \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 2(1 - \alpha)
\]

for some \( \alpha (0 \leq \alpha < 1) \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \), then \( f(z) \in S^*(\alpha) \).

**Theorem 1.8** If \( f(z) \in A \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j + 1 - 2\alpha)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \\
+ \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 2(1 - \alpha)
\]

for some \( \alpha (0 \leq \alpha < 1) \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \), then \( f(z) \in K(\alpha) \).

**Theorem 1.9** If \( f(z) \in A \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - \alpha + (1 - \alpha)e^{-2\lambda})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \\
+ \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 2(1 - \alpha) \cos \lambda
\]

for some \( \alpha (0 \leq \alpha < 1) \), \( \lambda (-\frac{\pi}{2} < \lambda < \frac{\pi}{2}) \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \), then \( f(z) \in S\mathcal{P}(\lambda, \alpha) \).
2 Properties of the class $S_b^*$

Generally, for some $b = 1 - \alpha$ or $b = (1 - \alpha)e^{-i\lambda} \cos \lambda \ (0 \leq \alpha < 1, -\frac{\pi}{2} < \lambda < \frac{\pi}{2})$, it is well known that $f(z) \in S_b^* = S^*(\alpha)$ or $SP(\lambda, \alpha)$ is univalent in $U$, but the almost $f(z) \in S_b^*$ is not univalent in $U$. Then, can we find the radius $r \ (0 < r \leq 1)$ for $f(z) \in S_b^*$ to be univalent in $U_r = \{ z \in \mathbb{C} : |z| < r \}$?

One of the answer of this problem about Koebe type function $f_b(z) \in S_b^*$ given by (1.3) is below.

**Theorem 2.1** $f_b(z) = \frac{z}{(1 - z)^{2b}} \in S_b^*$ belongs to the class $S^*$ in $U_r$, where

$$r = \left\{ \begin{array}{ll} \frac{|1 - b| - |b|}{1 - 2\text{Re}(b)} & \left( \text{Re}(b) \neq \frac{1}{2} \right) \\ \frac{1}{2|b|} & \left( \text{Re}(b) = \frac{1}{2} \right) \end{array} \right.$$

and $f_b(z)$ is also univalent in $U_r$.

**Proof.** Let us define

$$w = \frac{zf_b'(z)}{f_b(z)} = \frac{1 + (2b - 1)z}{1 - z}.$$

Then,

$$|z| = \left| \frac{w - 1}{w + (2b - 1)} \right| < r$$

or

$$\left| w - \frac{1 + r^2(2b - 1)}{1 - r^2} \right| < \frac{2|b|r}{1 - r^2}.$$

Therefore, finding the maximum of radius $r \ (0 < r \leq 1)$ satisfying the following inequality

$$\left( 2\text{Re}(b) - 1 \right) r^2 - 2|b|r + 1 \geq 0,$$

the proof is completed.

When $b \in \mathbb{R} \setminus \{0\}$, we obtain
Corollary 2.2 \( f_b(z) = \frac{z}{(1 - z)^{2b}} \in S^*_b \) belongs to the class \( S^* \) in \( U_r \), where

\[
r = \begin{cases} 
\frac{1}{1 - 2b} & (b < 0) \\
1 & (0 < b \leq 1) \\
\frac{1}{2b - 1} & (b > 1)
\end{cases}
\]

and \( f_b(z) \) is also univalent in \( U_r \).

Next, let \( F(z) = z \frac{f'(z)}{f(z)} = u + iv \) and \( b = \rho e^{i\varphi} \) \((\rho > 0, 0 \leq \varphi < 2\pi)\). Then, the condition of the definition of \( S^*_b \) is equivalent to

\[
(2.1) \quad \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] = 1 + \frac{\cos \varphi}{\rho} (u - 1) + \frac{\sin \varphi}{\rho} v > 0.
\]

Hence,

\[
u > \tan(-\varphi) v + 1 - \frac{\rho}{\cos \varphi} \quad (\cos \varphi > 0)
\]
or

\[
u < \tan(-\varphi) v + 1 - \frac{\rho}{\cos \varphi} \quad (\cos \varphi < 0).
\]

We denote by \( d(l_1, 1) \) the distance between the boundary line \( l_1 : (\cos \varphi)u + (\sin \varphi)v + \rho - \cos \varphi = 0 \) of the half plane satisfying the condition (2.1) and the point \( F(0) = 1 \). A simple computation gives us that

\[
d(l_1, 1) = \frac{|\cos \varphi \times 1 + \sin \varphi \times 0 + \rho - \cos \varphi|}{\sqrt{\cos^2 \varphi + \sin^2 \varphi}} = \rho,
\]

that is, that \( d(l_1, 1) \) is always equal to \( |b| = \rho \) regardless of \( \varphi \). Thus, if we consider the circle \( C \) with center at 1 and radius \( \rho \), then we can know the definition of \( S^*_b \) means that \( F(U) \) is covered by the half plane separated by a tangent line of \( C \) and containing \( C \).

By this fact, we consider the perpendicular \( l_2 : v = \tan \varphi (u - 1) \) of the line \( l_1 \) through the point 1, and also the intersection point \( z_0 \) of \( l_1 \) and \( l_2 \). And, we take the point \( \nu \) on \( l_2 \), satisfying the following relation

\[
(2.2) \quad |1 - z_0| \leq |\nu - z_0| \quad \text{and} \quad \arg(1 - z_0) = \arg(\nu - z_0).
\]
Namely, $\nu$ is farther from the point $z_0$ than the point 1 and in the side same as the point 1. Then, we see that if $F(U)$ is in the interior of the circle $C_{\nu}$ with center at $\nu$ and radius $|\nu - z_0|$, then $f(z)$ is a member of the class $S^*_b$.

Therefore, noting that the distance $d(1, \nu)$ between the point 1 and the point $\nu$ is $\left|\frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))}\right|$ (arg$(b) \neq \frac{\pi}{2}, \frac{3\pi}{2}$) or $|\text{Im}(\nu)|$ (arg$(b) = \frac{\pi}{2}$ or $\frac{3\pi}{2}$), we can derive the following two theorems.

**Theorem 2.3** If a function $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \left\{ |n - \nu| + |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right| a_n \right\} \leq |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right| - |1 - \nu|$$

for some $b \in \mathbb{C} \setminus \{0\}$ (arg$(b) \neq \frac{\pi}{2}, \frac{3\pi}{2}$) and a point $\nu$ given by (2.2), then $f(z) \in S^*_b$.

The proof is almost the same as that of Silverman’s result for Theorem 1.4.

**Proof.** It follows that if $f(z)$ satisfies

$$|zf'(z)f(z) - \nu| < |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right|,$$

then $f(z) \in S^*_b$, because $\frac{zf'(z)}{f(z)}$ maps $U$ onto the interior region of the circle $C_{\nu}$. Therefore, it is sufficient that we prove if $f(z)$ satisfies the inequality of the theorem, then the relation (2.3) holds true.

We know that

$$\left| \frac{zf'(z)}{f(z)} - \nu \right| = \left| \frac{(1 - \nu) + \sum_{n=2}^{\infty} (n - \nu)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right|$$

$$\leq \frac{|1 - \nu| + \sum_{n=2}^{\infty} (n - \nu)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n||z|^{n-1}}$$

$$< \frac{|1 - \nu| + \sum_{n=2}^{\infty} (n - 1)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}.$$
The upper bound of this last expression is $|b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right|$ if

$$|1 - \nu| + \sum_{n=2}^{\infty} (n - 1)|a_n| \leq \left( |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right| \right) \left( 1 - \sum_{n=2}^{\infty} |a_n| \right)$$

that is,

$$\sum_{n=2}^{\infty} \left( |n - \nu| + |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right| \right) |a_n| \leq |b| + \left| \frac{\text{Re}(\nu) - 1}{\cos(\text{arg}(b))} \right| - |1 - \nu|.$$  

Thus, the proof is completed.

\[ \Box \]

**Theorem 2.4**  If a function $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \{|n - \nu| + |b| + |\text{Im}(\nu)|\} |a_n| \leq |b|$$

for some $b \in \mathbb{C} \setminus \{0\}$ (arg$(b) = \frac{\pi}{2}$ or $\frac{3\pi}{2}$) and a point $\nu$ given by (2.2), then $f(z) \in S^*_i|b|$ or $f(z) \in S^*_{-i|b|}$.

**Proof.** If we use the same method of the proof of Theorem 2.3, then we can obtain

$$\sum_{n=2}^{\infty} \{|n - \nu| + |b| + |\text{Im}(\nu)|\} |a_n| \leq |b| + |\text{Im}(\nu)| - |1 - \nu|.$$  

Since $\nu = 1 + \text{Im}(\nu)$ when arg$(b) = \frac{\pi}{2}$ or arg$(b) = \frac{3\pi}{2}$, we have the inequality of the theorem. Hence, this completes the proof.

\[ \Box \]

**Corollary 2.5**  If a function $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \{|n - \nu| + |b| + |\nu - 1|\} |a_n| \leq |b|$$

for some $b \in \mathbb{C} \setminus \{0\}$ (arg$(b) = 0$ or $\pi$) and a point $\nu$ given by (2.2), then $f(z) \in S^*_i|b|$ or $f(z) \in S^*_{-i|b|}$.
Furthermore, taking $\nu = 1$ in Theorem 2.3 and Theorem 2.4, we know

**Corollary 2.6** If a function $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} (n-1+|b|)|a_n| \leq |b|$$

for some $b \in \mathbb{C} \setminus \{0\}$, then $f(z) \in \bigcap_{\varphi} S_{|b|e^{i\varphi}}^*$.

This result gives us the following fact.

**Remark 2.7** If a function $f(z) \in A$ satisfies Silverman’s inequality

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1-\alpha$$

for some $\alpha$ ($0 \leq \alpha < 1$), then $f(z)$ is not only starlike of order $\alpha$, also starlike of complex order $b$ for all $|b| = 1 - \alpha$. Namely, $f(z) \in \bigcap_{\varphi} S_{(1-\alpha)e^{i\varphi}}^*$.

And, letting $b = (1 - \alpha)e^{-i\lambda} \cos \lambda$ in Corollary 2.6, we deduce

**Corollary 2.8** If a function $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \left(n-1+(1-\alpha)\cos \lambda\right)|a_n| \leq (1-\alpha)\cos \lambda$$

for some $\alpha$ ($0 \leq \alpha < 1$) and $\lambda$ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$), then $f(z) \in SP(\lambda, \alpha)$, and in particular, $f(z) \in \bigcap_{\varphi} S_{(1-\alpha)e^{i\varphi}\cos \lambda}^*$.

### 3 Necessary conditions

In this section, we define the subclasses of $A$ as follows:

$$A(\theta) = \left\{ f(z) \in A : f(z) = z + \sum_{n=2}^{\infty} |a_n|e^{i(n-1)\theta+\pi} z^n \right\},$$

$$S_b^*(\theta) = A(\theta) \cap S_b^* \quad \text{and} \quad K_b(\theta) = A(\theta) \cap K_b$$
for some \( \theta \) \((0 \leq \theta < 2\pi)\). Then, we see that
\[
S^*_{1-\alpha}(0) = T^*(\alpha) \quad \text{and} \quad K_{1-\alpha}(0) = C(\alpha)
\]

\( T^*(\alpha) \) and \( C(\alpha) \) are introduced by Silverman [6].

Now, we discuss the necessary conditions for \( S^*_b(\theta) \) and \( K_b(\theta) \).

**Theorem 3.1** If a function \( f(z) \in S^*_b(\theta) \) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) \((0 \leq \theta < 2\pi)\), then the following inequality
\[
\sum_{n=2}^{\infty} \{(n - 1) \cos(\arg(b)) + |b|)|a_n| \leq |b|
\]
holds true.

**Proof.** By the definition of \( S^*_b(\theta) \), we can assume that
\[
\text{Re} \left[ \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > -1 \quad (z \in \mathbb{U})
\]
that is, that
\[
\text{Re} \left( e^{-i\arg(b)} \sum_{n=2}^{\infty} (n - 1)|a_n|e^{i((n-1)\theta+\pi)z^{n-1}} \right) > -|b| \quad (z \in \mathbb{U}).
\]

Next, letting \( z \to e^{-i\theta} \) along the values \( re^{-i\theta} \) \((0 \leq r < 1)\), we obtain
\[
\cos(\arg(b)) \frac{-\sum_{n=2}^{\infty} (n - 1)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|} \geq -|b|
\]
that is, that
\[
\sum_{n=2}^{\infty} \{(n - 1) \cos(\arg(b)) + |b|)|a_n| \leq |b|,
\]
which completes the proof of the theorem. \( \Box \)

Similarly, applying that \( f(z) \in K_b \) if and only if \( zf'(z) \in S^*_b \), we deduce
Theorem 3.2 If a function \( f(z) \in K_b(\theta) \) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta (0 \leq \theta < 2\pi) \), then the following inequality
\[
\sum_{n=2}^{\infty} \left( (n-1) \cos(\arg(b)) + |b| \right) |a_n| \leq |b|
\]
holds true.

By Corollary 2.6, the next remark is obtained.

Remark 3.3 For the case \( \cos(\arg(b)) = 1 \), \( f(z) \in S^*_b(\theta) \) if and only if
\[
\sum_{n=2}^{\infty} \left( (n-1) + b \right) |a_n| \leq b.
\]

When \( b \) is a pure imaginary value (\( b = i|b|, \cos(\arg(b)) = 0 \)), we know that

Corollary 3.4 If a function \( f(z) \) is in \( S^*_{i|b|}(\theta) \), then \( \sum_{n=2}^{\infty} |a_n| \leq 1 \) so that
\[
\text{Re} \left( \frac{f(z)}{z} \right) > 0.
\]

Corollary 3.5 If a function \( f(z) \) is in \( K_{i|b|}(\theta) \), then \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \) so that \( f(z) \in S^* \).

4 Distortion theorems

Next, applying Theorem 3.1 and Theorem 3.2, we consider the distortion theorems for the classes \( S^*_b(\theta) \) and \( K_b(\theta) \).

Theorem 4.1 If a function \( f(z) \in S^*_b(\theta) \) (\( \cos(\arg(b)) > 0 \)) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta (0 \leq \theta < 2\pi) \), then
\[
r - \delta_j - \frac{|b| - \lambda_j}{j \cos(\arg(b)) + |b|} r^{j+1} \leq |f(z)| \leq r + \delta_j + \frac{|b| - \lambda_j}{j \cos(\arg(b)) + |b|} r^{j+1} \quad (|z| = r < 1)
\]
where
\[
\delta_j = \begin{cases} 0 & (j = 1) \\ \frac{\sum_{n=2}^{j} |a_n| r^n}{j} & (j = 2, 3, 4, \ldots) \end{cases}
\]
and
\[
\lambda_j = \begin{cases} 
0 & (j = 1) \\
\sum_{n=2}^{j} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| & (j = 2, 3, 4, \ldots). 
\end{cases}
\]

**Proof.** By Theorem 3.1, it follows that if \( f(z) \in S^*_{\theta}(\theta) \), then
\[
\sum_{n=2}^{j} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| + (j \cos(\arg(b)) + |b|) \sum_{n=j+1}^{\infty} |a_n| \\
\leq \sum_{n=2}^{\infty} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| \leq |b|,
\]
or
\[
\sum_{n=j+1}^{\infty} |a_n| \leq \frac{|b| - \lambda_j}{j \cos(\arg(b)) + |b|} \quad (j = 1, 2, 3, \ldots).
\]

Therefore, we see that
\[
|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n|r^n = r + \sum_{n=2}^{j} |a_n|r^n + \sum_{n=j+1}^{\infty} |a_n|r^n \leq r + \delta_j + \frac{|b| - \lambda_j}{j \cos(\arg(b)) + |b|} r^{j+1}
\]
and
\[
|f(z)| \geq r - \sum_{n=2}^{\infty} |a_n|r^n = r - \sum_{n=2}^{j} |a_n|r^n - \sum_{n=j+1}^{\infty} |a_n|r^n \geq r - \delta_j - \frac{|b| - \lambda_j}{j \cos(\arg(b)) + |b|} r^{j+1}.
\]

This completes the proof of Theorem 4.1. \(\blacksquare\)

Making \( j = 1 \) in Theorem 4.1, we have

**Corollary 4.2** If a function \( f(z) \in S^*_{\theta}(\theta) \) \((\cos(\arg(b)) > 0)\) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) \((0 \leq \theta < 2\pi)\), then
\[
r - \frac{|b|}{\cos(\arg(b)) + |b|} r^2 \leq |f(z)| \leq r + \frac{|b|}{\cos(\arg(b)) + |b|} r^2 \quad (|z| = r < 1)
\]
with equality for \( f(z) = z - \frac{|b|}{\cos(\arg(b)) + |b|} e^{i\theta} \) \((z = \pm re^{-i\theta})\).
For the case \( \cos(\arg(b)) = 0 \) \((b = i|b|)\),

**Corollary 4.3** If a function \( f(z) \in S^*_{i|b|}(\theta) \) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) \((0 \leq \theta < 2\pi)\), then

\[
    r(1 - r) \leq |f(z)| \leq r(1 + r) \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - e^{i\theta}z^2 \) \((z = \pm re^{-i\theta})\).

When \( \cos(\arg(b)) = 1 \) \((b > 0)\),

**Corollary 4.4** If a function \( f(z) \in S^*_b(\theta) \) for some \( b > 0 \) and \( \theta \) \((0 \leq \theta < 2\pi)\), then

\[
    r - \frac{b}{1 + b}r^2 \leq |f(z)| \leq r + \frac{b}{1 + b}r^2 \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - \frac{b}{1 + b}e^{i\theta}z^2 \) \((z = \pm re^{-i\theta})\). In particular, for \( b = 1 - \alpha \) and \( \theta = 0 \), we obtain Theorem 4 in [6].

Using the same technique, we can discuss the similar theorem for \( K_b(\theta) \).

**Theorem 4.5** If a function \( f(z) \in K_b(\theta) \) \((\cos(\arg(b)) > 0)\) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) \((0 \leq \theta < 2\pi)\), then

\[
    r - \delta_j - \frac{|b| - \lambda_j^*}{(j + 1) \{ j \cos(\arg(b)) + |b| \}}r^{j+1}
\]

\[
    \leq |f(z)| \leq r + \delta_j + \frac{|b| - \lambda_j^*}{(j + 1) \{ j \cos(\arg(b)) + |b| \}}r^{j+1} \quad (|z| = r < 1),
\]

where

\[
    \delta_j = \left\{ \begin{array}{ll}
    0 & \quad (j = 1) \\
    \sum_{n=2}^{j} |a_n|r^n & \quad (j = 2, 3, 4, \ldots)
    \end{array} \right.
\]

and

\[
    \lambda_j^* = \left\{ \begin{array}{ll}
    0 & \quad (j = 1) \\
    \sum_{n=2}^{j} n \{ (n - 1) \cos(\arg(b)) + |b| \}|a_n| & \quad (j = 2, 3, 4, \ldots)
    \end{array} \right.
\]
Letting $j = 1$ in Theorem 4.5, we have

**Corollary 4.6** If a function $f(z) \in K_b(\theta)$ ($\cos(\arg(b)) > 0$) for some $b \in \mathbb{C} \setminus \{0\}$ and $\theta$ ($0 \leq \theta < 2\pi$), then

$$r - \frac{|b|}{2(\cos(\arg(b)) + |b|)} r^2 \leq |f(z)| \leq r + \frac{|b|}{2(\cos(\arg(b)) + |b|)} r^2 \quad (|z| = r < 1)$$

with equality for $f(z) = z - \frac{|b|}{2(\cos(\arg(b)) + |b|)} e^{i\theta} z^2$ ($z = \pm re^{-i\theta}$).

For $\cos(\arg(b)) = 0$ ($b = i|b|$),

**Corollary 4.7** If a function $f(z) \in K_{i|b|}(\theta)$ for some $b \in \mathbb{C} \setminus \{0\}$ and $\theta$ ($0 \leq \theta < 2\pi$), then

$$r \left(1 - \frac{1}{2}r\right) \leq |f(z)| \leq r \left(1 + \frac{1}{2}r\right) \quad (|z| = r < 1)$$

with equality for $f(z) = z - \frac{1}{2} e^{i\theta} z^2$ ($z = \pm re^{-i\theta}$).

When $\cos(\arg(b)) = 1$ ($b > 0$),

**Corollary 4.8** If a function $f(z) \in K_b(\theta)$ for some $b > 0$ and $\theta$ ($0 \leq \theta < 2\pi$), then

$$r - \frac{b}{2(1 + b)} r^2 \leq |f(z)| \leq r + \frac{b}{2(1 + b)} r^2 \quad (|z| = r < 1)$$

with equality for $f(z) = z - \frac{b}{2(1 + b)} e^{i\theta} z^2$ ($z = \pm re^{-i\theta}$). In particular, for $b = 1 - \alpha$ and $\theta = 0$, we obtain Corollary of Theorem 4 in [6].

Moreover, we also derive the following results.

**Theorem 4.9** If a function $f(z) \in S_b^*(\theta)$ ($\cos(\arg(b)) > 0$) for some $b \in \mathbb{C} \setminus \{0\}$ and $\theta$ ($0 \leq \theta < 2\pi$), then

$$1 - \delta^*_j - \frac{(j + 1)(|b| - \lambda_j)}{j \cos(\arg(b)) + |b|} r^j \leq |f'(z)| \leq 1 + \delta^*_j + \frac{(j + 1)(|b| - \lambda_j)}{j \cos(\arg(b)) + |b|} r^j \quad (|z| = r < 1),$$
where

\[
\delta_j^* = \begin{cases} 
0 & (j = 1) \\
\sum_{n=2}^{j} n|a_n|r^{n-1} & (j = 2, 3, 4, \cdots)
\end{cases}
\]

and

\[
\lambda_j = \begin{cases} 
0 & (j = 1) \\
\sum_{n=2}^{j} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| & (j = 2, 3, 4, \cdots).
\end{cases}
\]

\textbf{Proof.} Note that

\[
\sum_{n=2}^{\infty} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| \leq |b|
\]

implies that

\[
\sum_{n=2}^{j} \{(n - 1) \cos(\arg(b)) + |b|\} |a_n| + \frac{j \cos(\arg(b)) + |b|}{j + 1} \sum_{n=j+1}^{\infty} n|a_n| \leq |b|.
\]

This gives us that

\[
\sum_{n=j+1}^{\infty} n|a_n| \leq \frac{(j + 1)(|b| - \lambda_j)}{j \cos(\arg(b)) + |b|}.
\]

Therefore, we prove that

\[
|f'(z)| \leq 1 + \sum_{n=2}^{j} n|a_n|r^{n-1} + \sum_{n=j+1}^{\infty} n|a_n|r^{n-1} \leq 1 + \delta_j^* + r^j \sum_{n=j+1}^{\infty} n|a_n|
\]

\[
\leq 1 + \delta_j^* + \frac{(j + 1)(|b| - \lambda_j)}{j \cos(\arg(b)) + |b|} r^j
\]

and

\[
|f'(z)| \geq 1 - \delta_j^* - r^j \sum_{n=j+1}^{\infty} n|a_n| \geq 1 - \delta_j^* - \frac{(j + 1)(|b| - \lambda_j)}{j \cos(\arg(b)) + |b|} r^j.
\]

\qed
If we make \( j = 1 \) in Theorem 4.9, then we have

**Corollary 4.10** If a function \( f(z) \in S^*_b(\theta) \) (\( \cos(\arg(b)) > 0 \)) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) (\( 0 \leq \theta < 2\pi \)), then

\[
1 - \frac{2|b|}{\cos(\arg(b)) + |b|} r \leq |f'(z)| \leq 1 + \frac{2|b|}{\cos(\arg(b)) + |b|} r \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - \frac{|b|}{\cos(\arg(b)) + |b|} e^{i\theta} z^2 \) (\( z = \pm re^{-i\theta} \)).

When \( \cos(\arg(b)) = 1 \) (\( b > 0 \)),

**Corollary 4.11** If a function \( f(z) \in S^*_b(\theta) \) for some \( b > 0 \) and \( \theta \) (\( 0 \leq \theta < 2\pi \)), then

\[
1 - \frac{2b}{1+b} r \leq |f'(z)| \leq 1 + \frac{2b}{1+b} r \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - \frac{b}{1+b} e^{i\theta} z^2 \) (\( z = \pm re^{-i\theta} \)). In particular, for \( b = 1 - \alpha \) and \( \theta = 0 \), we obtain Theorem 6 in [6].

We also derive the following result for the class \( K_b(\theta) \) with Theorem 3.2.

**Theorem 4.12** If a function \( f(z) \in K_b(\theta) \) (\( \cos(\arg(b)) > 0 \)) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) (\( 0 \leq \theta < 2\pi \)), then

\[
1 - \delta^*_j - \frac{|b| - \lambda^*_j}{j \cos(\arg(b)) + |b|} r^j \leq |f'(z)| \leq 1 + \delta^*_j + \frac{|b| - \lambda^*_j}{j \cos(\arg(b)) + |b|} r^j \quad (|z| = r < 1),
\]

where

\[
\delta^*_j = \begin{cases} 
0 & (j = 1) \\
\sum_{n=2}^j n |a_n| r^{n-1} & (j = 2, 3, 4, \ldots)
\end{cases}
\]

and

\[
\lambda^*_j = \begin{cases} 
0 & (j = 1) \\
\sum_{n=2}^j n \{ (n - 1) \cos(\arg(b)) + |b| \} |a_n| & (j = 2, 3, 4, \ldots)
\end{cases}
\]
Letting \( j = 1 \) in Theorem 4.12, then we have

**Corollary 4.13** If a function \( f(z) \in \mathcal{K}_b(\theta) \) (\( \cos(\arg(b)) > 0 \)) for some \( b \in \mathbb{C} \setminus \{0\} \) and \( \theta \) (\( 0 \leq \theta < 2\pi \)), then

\[
1 - \frac{|b|}{\cos(\arg(b)) + |b|} r \leq |f'(z)| \leq 1 + \frac{|b|}{\cos(\arg(b)) + |b|} r \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - \frac{|b|}{2(\cos(\arg(b)) + |b|)} e^{i\theta} z^2 \) (\( z = \pm re^{-i\theta} \)).

When \( \cos(\arg(b)) = 1 \) (\( b > 0 \)),

**Corollary 4.14** If a function \( f(z) \in \mathcal{K}_b(\theta) \) for some \( b > 0 \) and \( \theta \) (\( 0 \leq \theta < 2\pi \)), then

\[
1 - \frac{b}{1+b} r \leq |f'(z)| \leq 1 + \frac{b}{1+b} r \quad (|z| = r < 1)
\]

with equality for \( f(z) = z - \frac{b}{2(1+b)} e^{i\theta} z^2 \) (\( z = \pm re^{-i\theta} \)). In particular, for \( b = 1 - \alpha \) and \( \theta = 0 \), we obtain Corollary of Theorem 6 in [6].

### 5 Applications of Carathéodory functions for \( S^*_b \) and \( \mathcal{K}_b \)

In this section, using Lemma 1.5, we obtain the extension of Corollary 2.6 with some interesting corollaries.

**Theorem 5.1** If a function \( f(z) \in \mathcal{A} \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \binom{k}{j-1} (1+2b)(-1)^{k-j} \left( \begin{array}{c} \beta \\ k-j \end{array} \right) a_j \left( \begin{array}{c} \gamma \\ n-k \end{array} \right) \right| + \left| \sum_{k=1}^{n} \binom{k}{j-1} (1)^{k-j} \left( \begin{array}{c} \beta \\ k-j \end{array} \right) a_j \left( \begin{array}{c} \gamma \\ n-k \end{array} \right) \right| \leq 2|b|
\]
for some \( b \in \mathbb{C} \setminus \{0\} \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \), then \( f(z) \in S_b^* \).

Proof. Let us define the function \( p(z) \) by

\[
p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right)
\]

for \( f(z) \in A \).

Applying Lemma 1.5, \( f(z) \in S_b^* \) if and only if

\[
(5.1) \quad p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U})
\]

for all \(|\zeta| = 1\).

Then, we need not consider Lemma 1.5 for \( z = 0 \), because it follows that

\[
p(0) = 1 \neq \frac{\zeta - 1}{\zeta + 1} \quad (|\zeta| = 1).
\]

Hence, the relation (5.1) is equivalent to

\[
(5.2) \quad 2bz + \sum_{n=2}^{\infty} \{(n - 1 + 2b) + \zeta(n - 1)\} a_n z^n \neq 0.
\]

Dividing the both sides of (5.2) by \( 2bz \) \( (z \neq 0) \), we obtain that

\[
(5.3) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0
\]

where

\[
A_n = \frac{(n - 1 + 2b) + \zeta(n - 1)}{2b} a_n \quad (n \geq 2).
\]

Therefore, multiplying the both sides of (5.3) by non-vanished function \( (1 - z)^\beta(1 + z)^\gamma \neq 0 \) in \( \mathbb{U} \), it is sufficient that we prove

\[
\left( 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \right) (1 - z)^\beta(1 + z)^\gamma = 1 + \sum_{n=2}^{\infty} \left\{ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \left( \begin{array}{c} \gamma \\ k-j \end{array} \right) a_j \left( \begin{array}{c} \delta \\ n-k \end{array} \right) \right\} \right\} z^{n-1} \neq 0
\]

where \( \beta, \gamma \in \mathbb{R} \) and \( A_1 = 1 \). Thus, if \( f(z) \) satisfies

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - 1 + 2b)(-1)^{k-j} \left( \begin{array}{c} \gamma \\ k-j \end{array} \right) a_j \right\} \left( \begin{array}{c} \delta \\ n-k \end{array} \right) \right| + |\zeta| \cdot \sum_{k=1}^{n} \left| \sum_{j=1}^{k} (j - 1)(-1)^{k-j} \left( \begin{array}{c} \gamma \\ k-j \end{array} \right) a_j \left( \begin{array}{c} \delta \\ n-k \end{array} \right) \right| \leq 2|b|
\]
then \(f(z) \in S_b^*\). The proof of Theorem 5.1 is completed.

We next derive the coefficient condition for functions \(f(z)\) to be in the class \(K_b\).

**Theorem 5.2** If a function \(f(z) \in A\) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left[ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j - 1 + 2b)(-1)^{k-j} \binom{\beta}{k-j}a_j \right\} \right] \left( \binom{\gamma}{n-k} \right) \leq 2|b|
\]

for some \(b \in \mathbb{C} \setminus \{0\}, \beta \in \mathbb{R}\) and \(\gamma \in \mathbb{R}\), then \(f(z) \in K_b\).

**Proof.** Since \(zf'(z) \in S_b^*\) if and only if \(f(z) \in K_b\) and since

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,
\]

replacing \(a_j\) in Theorem 5.1 by \(ja_j\), we easily prove Theorem 5.2.

Putting \(b = 1 - \alpha\) in Theorem 5.1 and Theorem 5.2, or \(b = (1 - \alpha)e^{-i\lambda} \cos \lambda\) in Theorem 5.1, we arrive Theorem 1.7, Theorem 1.8 and Theorem 1.9.

Furthermore, when \(\beta = \gamma = 0\) in Theorem 5.1 and Theorem 5.2, we obtain the following corollaries.

**Corollary 5.3** If a function \(f(z) \in A\) satisfies the following inequality

\[
\sum_{n=2}^{\infty} \left\{ |n - 1 + 2b| + (n - 1) \right\}|a_n| \leq 2|b|
\]

for some \(b \in \mathbb{C} \setminus \{0\}\), then \(f(z) \in S_b^*\).

**Corollary 5.4** If a function \(f(z) \in A\) satisfies the following inequality

\[
\sum_{n=2}^{\infty} n \left\{ |n - 1 + 2b| + (n - 1) \right\}|a_n| \leq 2|b|
\]
for some \( b \in \mathbb{C} \setminus \{0\} \), then \( f(z) \in K_b \).

Here, from the triangle inequality, it follows that
\[
\sum_{n=2}^{\infty} \{|n-1+2b|+(n-1)\}|a_n| \leq 2\sum_{n=2}^{\infty}(n-1+|b|)|a_n|.
\]

Thus, by this relation, we see that the inequality of Corollary 2.6 is stronger than that of Corollary 5.3. It shows that Corollary 5.3 is the improvement of Corollary 2.6 as the result for \( f(z) \in A \) to be in the class \( S^*_b \).

**Remark 5.5** For the case \( b > 0 \), the inequality of Corollary 5.3 is equivalent to that of Corollary 2.6.

**Remark 5.6** Setting \( b = -\frac{c}{2} \) \((c = 1, 2, 3, \ldots)\),

*Corollary 2.6* : \[
\sum_{n=2}^{\infty} \left( n - \frac{2-c}{2} \right) |a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S^*_{\frac{c}{2}+i\varphi} \subset S^*_c.
\]

*Corollary 5.3* : \[
\sum_{n=2}^{c} |a_n| + \frac{2}{c} \sum_{n=c+1}^{\infty} \left( n - \frac{2+c}{2} \right) |a_n| \leq 1 \implies f(z) \in S^*_c.
\]

**Remark 5.7** Taking \( b = -d \neq -\frac{c}{2} \) \((d > 0, \ c = 1, 2, 3, \ldots)\),

*Corollary 2.6* : \[
\sum_{n=2}^{\infty} \{|n-(1-d)|\}|a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S^*_{d+1+\varphi} \subset S^*_{-d} \cap S^*_d.
\]

In particular, if \( 0 < d \leq 1 \), then

*Corollary 2.6* : \[
\sum_{n=2}^{\infty} \{|n-(1-d)|\}|a_n| \leq 1 - (1-d) \implies f(z) \in \bigcap_{\varphi} S^*_{d+1+\varphi} \subset S^*_{-d} \cap S^*(1-d).
\]

*Corollary 5.3* : \[
\sum_{n=2}^{\lfloor 1+2d \rfloor} |a_n| + \frac{1}{d} \sum_{n=\lfloor 1+2d \rfloor+1}^{\infty} \{|n-(1+d)|\}|a_n| \leq 1 \implies f(z) \in S^*_d
\]

where the symbol \( [\ ] \) is defined by the following.

\( [x] \) is the integer and satisfies \( x - 1 < [x] \leq x \)

for all \( x \in \mathbb{R} \).
When some concrete values are substituted for $b$, the following corollary is obtained.

**Corollary 5.8**

For the case $b = -\frac{1}{2}$ $(c = 1)$,

Corollary 2.6: $2 \sum_{n=2}^{\infty} \left(n - \frac{1}{2}\right) |a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S^*_2 e^{i\varphi} \subset S^*_{-\frac{1}{2}} \cap S^*_{\frac{1}{2}}$.

Corollary 5.3: $2 \sum_{n=2}^{\infty} \left(n - \frac{3}{2}\right) |a_n| \leq 1 \implies f(z) \in S^*_{-\frac{1}{2}}$.

For the case $b = -1$ $(c = 2)$,

Corollary 2.6: $\sum_{n=2}^{\infty} n |a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S^*_e e^{i\varphi} \subset S^*_1 \cap S^*$.

Corollary 5.3: $|a_2| + \sum_{n=3}^{\infty} (n - 2) |a_n| \leq 1 \implies f(z) \in S^*_1$.

For the case $b = -\frac{3}{2}$ $(c = 3)$,

Corollary 2.6: $\frac{2}{3} \sum_{n=2}^{\infty} \left(n + \frac{1}{2}\right) |a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S^*_2 e^{i\varphi} \subset S^*_{-\frac{3}{2}}$.

Corollary 5.3: $|a_2| + |a_3| + \frac{2}{3} \sum_{n=4}^{\infty} \left(n - \frac{5}{2}\right) |a_n| \leq 1 \implies f(z) \in S^*_{-\frac{3}{2}}$. 
For the case $b = -2$ \hspace{1em} (c = 4),

Corollary 2.6 : \quad \frac{1}{2} \sum_{n=2}^{\infty} (n + 1)|a_n| \leq 1 \implies f(z) \in \bigcap_{\varphi} S_{2e^{i\varphi}}^* \subset \mathcal{S}_{-2}^*.

Corollary 5.3 : \quad |a_2| + |a_3| + |a_4| + \frac{1}{2} \sum_{n=5}^{\infty} (n - 3)|a_n| \leq 1 \implies f(z) \in \mathcal{S}_{-2}^*.

**Example 5.9** For $|\varepsilon| \leq \frac{1}{2}$, a function $f(z) = z + \varepsilon z^2$ satisfies the inequality of Corollary 2.6 with $b = -1$, that is, $f(z) \in \mathcal{S}^*$ and for $|\varepsilon| \leq 1$, satisfies that of Corollary 5.3 with $b = -1$, that is, $f(z) \in \mathcal{S}_{-1}^*$.

Actually, it follows that

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) = \text{Re} \left( 2 - \frac{1}{1 + \varepsilon z} \right) > 2 - \frac{1}{1 - |\varepsilon|} = \frac{1 - 2|\varepsilon|}{1 - |\varepsilon|} \geq 0 \quad \text{for all } |\varepsilon| \leq \frac{1}{2}
\]

and

\[
\text{Re} \left[ 1 + \frac{1}{-1} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] = \text{Re} \left( \frac{1}{1 + \varepsilon z} \right) > \frac{1}{1 + |\varepsilon|} > 0 \quad \text{for all } |\varepsilon| \leq 1.
\]

Thus, $f(z) \in \mathcal{S}^* \hspace{1em} (|\varepsilon| \leq \frac{1}{2})$ and $f(z) \in \mathcal{S}_{-1}^* \hspace{1em} (|\varepsilon| \leq 1)$.

Finally, the observation of the proof of Theorem 5.2, we can derive the similar results for the class $\mathcal{K}_b$.

**References**


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